FIRST AND SECOND VARIATIONS OF HYPERSURFACE VOLUME

We consider an oriented hypersurface $\Sigma^{n-1} \subset M^n$ in an oriented Riemannian manifold (M, g), endowing Σ with the induced Riemannian metric and Levi-Civita connection. Σ is not assumed compact, and indeed we consider a bounded domain $\Omega \subset \Sigma$. The variation vector field X ('velocity vector') is not assumed normal to Σ or with compact support in Ω, and Σ is not assumed minimal. Thus there will be boundary terms and terms involving the tangential component X^T and mean curvature H. (*Source:* We combine elements from D. Lee's book and E. Kuwert's thesis, while aiming to slightly simplify the calculation.)

1. Linear Algebra Review.

1. Let (E, g) be an n-dimensional real vector space, endowed with a choice of orientation and a positive-definite inner product g; let (e_i) be a g-orthonormal positive frame, $\theta_i \in E^*$ the dual co-frame. The g-volume form on E is:

$$\omega_E = \theta_1 \wedge \ldots \wedge \theta_n \in \Lambda^n E^*.$$

Given an $(n-1)$-dimensional subspace $V \subset E$, we consider the induced volume form $\omega_V \in \Lambda^{n-1} V^*$, which can be defined as follows: if $v_1, \ldots, v_{n-1}$ are vectors in V and ν is a choice of unit normal to V in E (defining the induced orientation in V), let:

$$\omega_V(v_1, \ldots, v_{n-1}) = \det(A), \quad A = [v_1| \ldots |v_{n-1}|\nu], \quad \text{(by columns)} .$$

Here the columns of the $n \times n$ matrix A are vectors in R^n, obtained by expressing each v_i and ν in the basis (e_i). One easily sees that the $n \times n$ matrix $A^t A$ is in 'block form': an $(n-1) \times (n-1)$ block with entries $\langle v_i, v_j \rangle_g$ and the n, n entry $|\nu|_g^2 = 1$ (the other entries are zero). Thus, if we let $G_{ij} = \langle v_i, v_j \rangle_g$ (an $(n-1) \times (n-1)$ matrix), we have:

$$\det G = \det A^t A = (\det A)^2,$$

and thus (assuming the (v_i) are either linearly dependent, or a positive basis of V to take the positive square root):

$$\omega_V(v_1, \ldots, v_{n-1}) = \sqrt{\det \langle v_i, v_j \rangle_g}, \quad v_i \in V \subset E.$$

2. Let $A(t)$ be a C^1 curve in GL_n^+ ($n \times n$ matrices with positive determinant). We have the formula:

$$(\det A(t))' = tr(A' A^{-1}) \det A(t).$$

To see this, recall the Gram-Schmidt process defines a factorization $A(t) = U(t)T(t)$, with $U(t) \in SO_n$ and $T(t) \in B_n^+$ (group of upper-triangular matrices, with zeros below the diagonal and positive diagonal entries.) We have $\det A(t) = \det T(t)$, and thus:

$$(\det A(t))' = (\det T(t))' = (\frac{t'_{11}}{t_{11}} + \ldots + \frac{t'_{nn}}{t_{nn}})(t_{11}(t) \ldots t_{nn}(t)) = tr(T'T^{-1}) \det A(t).$$

On the other hand, from $T = U^{-1}A = U^tA'$ and $T^{-1} = A^{-1}U$, we compute:

$$T' = (U^t)'A + U^tA, \quad T'T^{-1} = (U^t)'U + U^tA'A^{-1}U.$$

Since $U^tU = \mathbb{I}_n$, we have $(U^t)'U + U^tU' = 0$, and hence $2tr((U^t)'U) = tr[(U^t)'U + U^tU'] = 0$. We conclude $tr(T'T^{-1}) = tr(U^tA'A^{-1}U) = tr(A'A^{-1})$, as we wished to show. $\square$

3. In particular, if $A(t)$ is a C^1 curve in GL_n^+ with $A(0) = \mathbb{I}_n$, we have $(\det A)'(0) = tr(A'(0))$, and compute the second derivative:

$$(\det A)'' = \{tr(A''A^{-1} - (A')^2A^{-2}) + [tr(A'A^{-1})]^2\} \det A(t)$$

$$= tr(A''(0)) - tr(A'(0))^2 + (tr(A'(0))^2$$

at $t = 0$. Now, if $A(t)$ is symmetric, $tr(A'(0)^2) = \sum_{i,j} A'(0)_{ij}^2 = |A'(0)|^2$, so we conclude:

$$(\det A)''(0) = tr(A''(0)) - |A'(0)|^2 + (tr(A'(0))^2.$$

4. We are interested in the first and second derivatives of $J(t) = \sqrt{\det\langle \bar{e}_i, \bar{e}_j\rangle_g}$ at $t = 0$. So set $A_{ij}(t) = \langle \bar{e}_i, \bar{e}_j\rangle_g$, with $A(t) \in GL_n^+$ starting at $A(0) = \mathbb{I}_n$ and $J^2(t) = \det A(t)$, $J(0) = 1$. Then starting from:

$$(J^2)' = 2JJ', \quad (J^2)'' = 2JJ'' + 2(J')^2,$$

we easily obtain:

$$J'(0) = \frac{1}{2}tr(A'(0)), \quad J''(0) = \frac{1}{2}[tr(A''(0)) - |A'(0)|^2 + \frac{1}{2}(tr(A'(0))^2].$$

2. Setting up the calculation.

We consider a bounded domain $\Omega \subset \Sigma$, with unit outward normal $\eta \in T\Sigma$ ('conormal'), while ν denotes the unit normal of Σ in M, defining its orientation. D_XY is the covariant derivative in M, its tangential component D_X^TY (when $X, Y \in T\Sigma$) the covariant derivative for the Levi-Civita connection of the induced metric on Σ. The scalar second fundamental form is $A(X,Y) = \langle D_X\nu, Y\rangle$, and the mean curvature of Σ in M is its trace, $H = \sum_i A(e_i, e_i)$.

Typically one starts from a variation (F_t) of Σ: a one-parameter family of embeddings $F_t : \Sigma \to M$, with F_0 the inclusion map, also written $F(q,t)$ ($q \in \Sigma, t \in I$, an open interval containing 0). Then set $\bar{X}(q,t) = \partial_tF(q,t)$, a 'vector field along F', meaning $\bar{X}(q,t) \in T_{F(q,t)}M$ (definitely not the same as a vector field on M, defined in a neighborhood of Σ.) This restricts, when $t = 0$, to a vector field $X \in TM_{|\Sigma}, X(q) = \bar{X}(q, 0)$.

Alternatively, one may start from a vector field $X \in TM_{|\Sigma}$, extend it to a vector field $\bar{X}$ in a neighborhood of Σ, and let (F_t) be the local flow of $\bar{X}$, a one-parameter group of embeddings $F_t : \Sigma \to M$, $t \in I$, F_0 the inclusion. Some terms in the second variation formula depend on the extension chosen. These points of view are not equivalent; the first one is more general, so we'll adopt it, taking the variation (F_t) as given.

We are interested in the rates of change of volume of the sets $\Omega_t = F_t(\Omega) \subset \Sigma_t = F_t(\Sigma)$. The volume form ω_{Σ_t} induced from ω_M and the unit normal to Σ_t is associated, at $q_t = F(q,t)$, with the $(n-1)$-dimensional subspace $T_{q_t}\Sigma_t = dF_t(q)[T_q\Sigma]$. We have:

$$vol(\Omega_t) = \int_\Omega F_t^*\omega_{\Sigma_t} = \int_\Omega J(q,t)\omega_\Sigma.$$

In the above expression, with $\omega_M \in \Omega^n(M), \omega_\Sigma \in \Omega^{n-1}(\Sigma)$ the Riemannian volume forms on M and Σ, we consider:

$$\omega_t = F_t^*\omega_{\Sigma_t} \in \Omega^{n-1}(\Sigma), \quad \omega_t(q,t) = J(q,t)\omega_\Sigma(q),$$

and the first and second partial derivatives of the Jacobian function J at $t = 0$ and points $q \in \Sigma$:

$$a_X(p) = \frac{\partial J}{\partial t}(q,0), \quad b_X(q) = \frac{\partial^2 J}{\partial t^2}(q,0).$$

These are the integrands (over Σ) in the first and second variations of hypersurface volume.

To calculate in local coordinates, consider a local chart $\varphi : U_0 \to U$, $\varphi(0) = p$, $U_0 \subset R^{n-1}, U \subset \Sigma$. We may choose φ so that $e_i(x) = \partial_{x_i}\varphi(x) \in T_q\Sigma, q = \varphi(x)$, defines a positive orthonormal frame at the point $\varphi(0) = p \in U$. (The e_i are 'vector fields along φ'; but since φ is a local chart, we may think of them as tangent vector fields in $U \subset \Sigma$.) In fact, choosing exponential normal coordinates based at a fixed point $p \in \Sigma$, we may also assume that $D_v^T e_i(p) = 0$, for any $v \in T_p\Sigma$. From the local chart φ and the variation (F_t) we define a smooth map (not always an immersion):

$$\Phi : U_0 \times I \to M, \quad \Phi(x,t) = F_t(\varphi(x)), \quad \Phi_t = F_t \circ \varphi : U_0 \to \Sigma_t.$$

Extending the previously used notation, we set:

$$\bar{X}(x,t) = \partial_t\Phi(x,t), \quad \bar{e}_i(x,t) = \partial_{x_i}\Phi(x,t) = dF_t(\varphi(x))[e_i(x)].$$

vector fields along Φ on M (sections of the pullback of the tangent bundle). (And $X(x) = \bar{X}(x,0)$.) Of course the $\bar{e}_i$ are not in general orthonormal, except at $x = 0, t = 0$. We have the well-known relation:

$$\frac{D}{\partial t}\frac{\partial \Phi}{\partial x_i}(x,t) = \frac{D}{\partial x_i}\frac{\partial \Phi}{\partial t}(x,t).$$

This may be written in the form:

$$(D_{\bar{X}}\bar{e}_i)(x,t) = (D_{\bar{e}_i}\bar{X})(x,t),$$

as an equality (in $U_0 \times I$) of vector fields along Φ. As for the volume under variation, we have:

$$\Phi_t^*\omega_{\Sigma_t} = J(x,t)d^{n-1}x, \quad vol(F_t(U)) = \int_{U_0} J(x,t)d^{n-1}x.$$

where $d^{n-1}x$ is the volume form in $U_0 \subset R^{n-1}$ and $J(x,t) > 0$ is the Jacobian. We are interested in computing the first and second variation integrands (at $x = 0$ in U_0, corresponding to $p \in \Sigma$):

$$a_X(0) = \frac{\partial J}{\partial t}(0,0), \quad b_X(0) = \frac{\partial^2 J}{\partial t^2}(0,0),$$

where:

$$J(x,t) = \omega_{\Sigma_t}(\bar{e}_1(x,t), \dots, \bar{e}_{n-1}(x,t)) = \sqrt{\det\langle \bar{e}_i(x,t), \bar{e}_j(x,t)\rangle_{\Phi^* g}}$$

as seen in (1.) of the previous section. Note the $\bar{e}_i(x,t)$ define a positive basis of the subspace $T_{x_t}\Sigma_t = dF_t(x)[T_x\Sigma] \subset T_{x_t}M, x_t = F_t(x)$. (We henceforth identify $\Phi^* g$ and g in the notation.)

Remark. It is incorrect to think of the $\bar{e}_i$ as vector fields on M, defined in a neighborhood of Σ, and extending the frame e_i on Σ. Here is the problem: from $[\bar{X}, \bar{e}_i]_M(x) = 0$ (which follows from $\bar{e}_i(x,t) = dF_t(x)[e_i(x)]$), a short calculation shows that, on Σ (taking tangential and normal components, where $X = X^T + \phi\nu$ and $S : T\Sigma \to T\Sigma$ is the Weingarten operator):

$$[X^T, e_i]_\Sigma = \phi(Se_i - (D_\nu \bar{e}_i)^T), \quad e_i(\phi)\nu = \phi(D_\nu \bar{e}_i)^\perp,$$

leading to the compatibility conditions:

$$\phi(x) = 0 \Rightarrow \nabla^\Sigma \phi(x) = 0 \text{ and } [X^T, e_i]_\Sigma(x) = 0,$$

which are of course not assumed.

3. The first variation formula.

With $A_{ij}(x,t) = \langle \bar{e}_i(x,t), \bar{e}_j(x,t)\rangle_g$, we have:

$$\frac{\partial A_{ij}}{\partial t} = \langle D_{\bar{X}}\bar{e}_i, \bar{e}_j\rangle + \langle \bar{e}_i, D_{\bar{X}}\bar{e}_j\rangle = \langle D_{\bar{e}_i}\bar{X}, \bar{e}_j\rangle + \langle \bar{e}_i, D_{\bar{e}_j}\bar{X}\rangle$$

Thus:

$$\frac{1}{2}tr(\frac{\partial A}{\partial t})(x,t)=\sum_i\langle D_{\bar e_i}\bar X,\bar e_i\rangle.$$

Evaluate at $t=0$, introducing the decomposition $X=X^T+\phi\nu$, where $X^T\in T\Sigma$ and $\phi:\Omega\to R$:

$$a_X(x)=\frac{\partial J}{\partial t}_{|t=0}(x)=\frac{1}{2}tr(\frac{\partial A}{\partial t})(x,0)=\sum_i\langle D_{e_i}X,e_i\rangle$$

$$=\phi\sum_i\langle D^T_{e_i}\nu,e_i\rangle+\sum_i\langle D^T_{e_i}X^T,e_i\rangle=\phi H+div_\Sigma X^T.$$

In integrated form, using the divergence theorem, we have the well-known expression:

$$\frac{d}{dt}vol(f_t(\Omega))_{|t=0}=\int_\Omega a_X d\mu_\Sigma=\int_\Omega\phi H d\mu_\Sigma+\int_{\partial\Omega}\langle X,\eta\rangle d\mu_{\partial\Omega}.$$

4. The second variation, Part I: derivation of the index integrand.

We begin by computing the second derivative of A_{ij}:

$$\frac{\partial^2}{\partial t^2}\langle\bar e_i(x,t),\bar e_j(x,t)\rangle=\frac{\partial}{\partial t}\langle D_{\bar e_i}\bar X,\bar e_j\rangle+(i\leftrightarrow j)$$
$$=\langle D_{\bar X}D_{\bar e_i}\bar X,\bar e_j\rangle+\langle D_{\bar e_i}\bar X,D_{\bar X}\bar e_j\rangle$$
$$=\langle D_{\bar e_i}D_{\bar X}\bar X,\bar e_j\rangle+R^M(\bar X,\bar e_i,\bar X,\bar e_j)+\langle D_{\bar e_i}\bar X,D_{\bar e_j}\bar X\rangle+(i\leftrightarrow j).$$

Define $Z=D_{\bar X}\bar X\in TM$, the 'acceleration vector field' of the variation, with values in TM. (More precisely, a vector field along Φ, or section of Φ^*TM.)

Setting $t=0$, we have:

$$\frac{\partial^2 A_{ij}}{\partial t^2}(x,0)=\langle D_{e_i}Z,e_j\rangle+\langle D_{e_j}Z,e_i\rangle+2\langle D_{e_i}X,D_{e_j}X\rangle-2R^M(e_i,X,X,e_j).$$

Taking traces:

$$\frac{\partial^2 tr(A)}{\partial t^2}(x,0)=2div_\Sigma Z+2|\sum_i D_{e_i}X|^2-2\sum_i R^M(e_i,X,X,e_i).$$

This is the first of three terms (see part 1, no.4) used in the computation of $b_X(x)=\frac{\partial^2 J}{\partial t^2}(x,0)$. The other two terms are:

$$|\frac{\partial A}{\partial t}(x,0)|^2=\sum_{i,j}(\langle D_{e_i}X,e_j\rangle+\langle D_{e_j}X,e_i\rangle)^2=2\sum_i|D^T_{e_i}X|^2+2\sum_{i,j}\langle D_{e_i}X,e_j\rangle\langle D_{e_j}X,e_i\rangle$$

$$(\frac{\partial tr(A)}{\partial t})^2(x,0)=4(div_\Sigma X)^2.$$

Thus we have for the second variation integrand:

$$b_X(x) = \frac{\partial J}{\partial t}(x,0) = \frac{1}{2}\frac{\partial^2 tr(A)}{\partial t^2}(x,0) - \frac{1}{2}|\frac{\partial A}{\partial t}(x,0)|^2 + \frac{1}{4}(\frac{\partial tr(A)}{\partial t})^2(x,0)$$

$$= div_\Sigma Z + (div_\Sigma X)^2 + \sum_i |(D_{e_i}X)^\perp|^2 - \sum_{ij}\langle D_{e_i}X, e_j\rangle\langle D_{e_j}X, e_i\rangle - \sum_i R^M(e_i, X, X, e_i).$$

The divergence of the acceleration field Z contributes only a boundary integral, so we write this in the form:

$$b_X = div_\Sigma Z + I(X,X),$$

where the 'index integrand' $I(X,Y)$ is defined as:

$$I(X,Y) = (div_\Sigma X)(div_\Sigma Y) + \sum_i \langle (D_{e_i}X)^\perp, (D_{e_i}Y)^\perp\rangle - \sum_{i,j}\langle D_{e_i}X, e_j\rangle\langle D_{e_j}Y, e_i\rangle - \sum_i R^M(e_i, X, Y, e_i).$$

Bearing in mind the decomposition $X = X^T + \phi\nu$, the plan now is to understand separately the terms $I(\phi\nu,\phi\nu), I(X^T,\phi\nu)$ and $I(X^T,X^T)$.

A. The index integrand on normal-normal terms.

This is quick, observing $div_\Sigma(\phi\nu) = \phi(div_\Sigma\nu) = \phi H$, $\langle D_{e_i}(\phi\nu), e_j\rangle = \phi\langle D_{e_i}\nu, e_j\rangle = \phi A(e_i,e_j)$ and $D_{e_i}(\phi\nu))^\perp = e_i(\phi)\nu$. We find:

$$I(\phi\nu,\phi\nu) = |\nabla^\Sigma\phi|^2 - (Ric^M(\nu,\nu) + |A|^2 - H^2)\phi^2,$$

the classical formula for minimal surfaces, with the added term H^2 in this more general case.

B. The index integrand on tangential-normal terms. $I(X,\phi\nu), X \in T\Sigma$. We have:

$$\sum_i \langle (D_{e_i}X)^\perp, (D_{e_i}(\phi\nu))^\perp\rangle = \sum_i \langle (D_{e_i}X)^\perp, e_i(\phi)\nu\rangle = \langle D_{\nabla^\Sigma\phi}X, \nu\rangle = -A(X, \nabla^\Sigma\phi).$$

$$\sum_{i,j}\langle D_{e_i}X, e_j\rangle\langle D_{e_j}(\phi\nu), e_i\rangle = \sum_{i,j}\langle D_{e_i}X, e_j\rangle\phi A(e_i,e_j) = \phi\sum_i \langle D^\Sigma_{e_i}X, S(e_i)\rangle.$$

The other two terms are $(div_\Sigma X)\phi H$ and the curvature term, both linear in ϕ.

We seek to combine the terms which do not involve $\nabla^\Sigma\phi$ into a divergence term. Recall the Codazzi equation:

$$(D^\Sigma_X S)Y - (D^\Sigma_Y S)X = R^M(X,Y)\nu, \quad X, Y \in T\Sigma.$$

Thus, at $x = 0$ (corresponding to $\varphi(0) = p \in \Sigma$ under the local chart φ for Σ):

$$\sum_i \langle Se_i, D^\Sigma_{e_i}X\rangle + \sum_i R^M(e_i, X, \nu, e_i) = \sum_i \langle Se_i, D^\Sigma_{e_i}X\rangle + \langle (D^\Sigma_{e_i}S)X, e_i\rangle - \langle (D^\Sigma_X S)e_i, e_i\rangle$$

$$= \sum_i [\langle (D^{\Sigma}_{e_i}(SX), e_i\rangle - X\langle Se_i, e_i\rangle = div_{\Sigma}(SX) - X(H),$$

(using $D^T_X e_i(x) = 0$ at the given point $x = 0, \varphi(x) = p$ for X tangential, as we may assume.) Combining these facts, we find:

$$I(X, \phi\nu) = (div_{\Sigma} X)(\phi H) - A(X, \nabla^{\Sigma}\phi) - \phi(div_{\Sigma}(SX) - X(H))$$
$$= \phi(H div_{\Sigma} X + X(H) - div_{\Sigma}(SX)) - A(X, \nabla^{\Sigma}\phi)$$
$$= \phi(div_{\Sigma}(HX) - div_{\Sigma}(SX)) - A(X, \nabla^{\Sigma}\phi).$$

We can also write this in the form of [Lee, p.36], noting that:

$$div_{\Sigma}(H\phi X) - HX(\phi) = \phi div_{\Sigma}(HX), \qquad div_{\Sigma}(\phi SX) = \phi div_{\Sigma}(SX) + A(X, \nabla^{\Sigma}\phi).$$

We conclude:

$$I(X, \phi\nu) = div_{\Sigma}(H\phi X - \phi SX) - HX(\phi),$$

for X tangential.

C. The index integrand on tangential-tangential terms. We compute $I(X, X)$, assuming $X \in T\Sigma$.

$$\sum_i |(D_{e_i} X)^{\perp}|^2 = \sum_i |\vec{A}(e_i, X)|^2 = |S(X)|^2.$$

And from the Gauss formula:

$$\sum_i R^M(e_i, X, X, e_i) = Ric^{\Sigma}(X, X) + |S(X)|^2 - HA(X, X).$$

As observed earlier, since $D_{\bar{e}_i}\bar{X} = D_{\bar{X}}\bar{e}_i$ (as vector fields along Φ), we have at points $\Phi(x, 0) = \varphi(x) \in U \subset \Sigma$: $D_{e_i} X = D_X \bar{e}_i$, and if X is tangential in U_0: $D_{e_i} X = D_X e_i$ on Σ, in particular for the tangential components: $D^T_{e_i} X = D^T_X e_i$ on Σ.

On the other hand, we may assume at the point $p = \varphi(0) \in \Sigma$ of calculation we have $D^T_v e_i(p) = 0$, for any $v \in T_p\Sigma$ and all i. Thus the term involving the sum over i, j of $\langle D_{e_i} X, e_j\rangle\langle D_{e_j} X, e_i\rangle$ does not contribute to index integrand, at the point p.

After cancelation of the term $|S(X)|^2$, we are left with:

$$I(X, X) = (div_{\Sigma} X)^2 - Ric^{\Sigma}(X, X) + HA(X, X).$$

We expect this will turn out to be almost entirely a divergence term. So compute:

$$div_{\Sigma}((div_{\Sigma} X)X = (div_{\Sigma} X)^2 + X(div_{\Sigma} X).$$

And again using the fact the frame (e_i) is parallel at $x = 0$:

$$X(div_{\Sigma} X) = \sum_i X\langle D^T_{e_i} X, e_i\rangle = \sum_i \langle D^T_X D^T_{e_i} X, e_i\rangle$$

$$= \sum_i [\langle D^T_{e_i} D^T_X X, e_i\rangle + R^\Sigma(X, e_i, X, e_i)] = div_\Sigma(D^T_X X) - Ric^\Sigma(X, X),$$

and thus:

$$(div_\Sigma X)^2 = div_\Sigma((div_\Sigma X)X) - div_\Sigma(D^T_X X) + Ric^\Sigma(X, X),$$

giving, at the point $p = \varphi(0)$:

$$I(X, X) = HA(X, X) + div_\Sigma[(div_\Sigma X)X - D^T_X X].$$

D. Putting everything together.

For a general variation vector field $X = X^T + \phi\nu$, using the fact the index integrand is linear over R (not over functions!) we have:

$$I(X, X) = I(\phi\nu, \phi\nu) + 2I(X^T, \phi\nu) + I(X^T, X^T)$$

$$= |\nabla^\Sigma \phi|^2 - (Ric^M(\nu, \nu) + |A|^2 - H^2)\phi^2 - 2HX^T(\phi) + HA(X^T, X^T)$$
$$+ div_\Sigma[2H\phi X^T - 2\phi S(X^T) + (div_\Sigma X^T)X^T - D^T_{X^T} X^T]$$

To compute the term $div_\Sigma Z$ in b_X, we let $Z = Z^T + \zeta\nu$, $Z^T \in T\Sigma$, and find:

$$\sum_i \langle D_{e_i} Z, e_i\rangle = \sum_i \langle D^T_{e_i} Z^T, e_i\rangle + \zeta H = div_\Sigma Z^T + \zeta H.$$

So the final expression for the second variation integrand is:

$$b_X = \zeta H + |\nabla^\Sigma\phi|^2 - (Ric^M(\nu, \nu) + |A|^2 - H^2)\phi^2 - 2HX^T(\phi) + HA(X^T, X^T) + div_\Sigma(c_X),$$

$$c_X = Z^T + 2H\phi X^T - 2\phi S(X^T) + (div_\Sigma X^T)X^T - D^T_{X^T} X^T.$$

The integrated general second variation formula reads:

$$\frac{d^2 vol(f_t(\Omega))}{dt^2}_{|t=0} = \int_\Omega \{|\nabla^\Sigma\phi|^2 - (Ric^M(\nu, \nu) + |A|^2 - H^2)\phi^2$$
$$+ H[\zeta - 2X^T(\phi) + A(X^T, X^T)]\} d\mu_\Sigma + \int_{\partial\Omega} \langle c_X, \eta\rangle d\mu_{\partial\Omega}.$$

We see that in the minimal surface case, the acceleration vector field and tangential components of X contribute only boundary terms.

Remark on the acceleration vector.

We may get some control on the terms of the decomposition $Z = Z^T + \zeta\nu$ on Σ if we take the point of view that X is given on Σ, extended to $\bar{X}$ in a neighborhood of Σ, and (F_t) is the local flow of $\bar{X}$, and take a particular extension of X: extend ν to a neighborhood of Σ as a geodesic vector field $\bar{\nu}$, that is, $D_{\bar{\nu}}\bar{\nu} = 0$, then extend ϕ to be constant along normal geodesics ($\bar{\nu}(\bar{\phi}) = 0$)

and extend $\bar{X}^T$ by parallel transport: $D_{\bar{\nu}}\bar{X}^T = 0$. Finally, set $\bar{X} = \bar{X}^T + \bar{\phi}\bar{\nu}$, extending the decomposition $X = X^T + \phi\nu$ on Σ. Then a simple calculation yields, on Σ:

$$Z^T = D^T_{X^T} X^T + \phi S(X^T), \quad \zeta = -A(X^T, X^T) + X^T(\phi).$$

This leads to a simplification of terms in b_X and c_X:

$$b_X = |\nabla^\Sigma \phi|^2 - (Ric^M(\nu,\nu) + |A|^2 - H^2)\phi^2 - HX^T(\phi) + div_\Sigma(c_X),$$
$$c_X = 2H\phi X^T - \phi S(X^T) + (div_\Sigma X^T)X^T.$$

Example: scalar curvature of a rotationally symmetric metric.
The usual form of a rotationally symmetric metric in R^n is:

$$g = ds^2 + r^2(s)d\omega^2,$$

where $d\omega^2$ is the standard metric in S^{n-1}. Thus the $(n-1)$-dimensional volume ('area') of the sphere $\{s = s_0\}$ in the metric g is $\omega_{n-1}r(s)^{n-1}$, and $\frac{dr}{ds}(s_0)$ corresponds to a minimal surface at s_0. If no such minimal surface exists, r is monotone in s, and we can instead use r as a parameter (the 'area radius') and write the metric in the form:

$$g = \frac{dr^2}{V(r)} + r^2 d\omega^2, \quad \frac{dr}{ds} = \sqrt{V(r)}.$$

We can use the first and second variation formulas to compute the scalar curvature R_g of g. The volume $|S_r|$ at radius r is $\omega_{n-1}r^{n-1}$, and $\sqrt{V}\partial_r$ is a unit normal vector, so first variation gives:

$$\sqrt{V}\partial_r|S_r| = \int_{S_r} Hr^{n-1}d\omega, \text{ or } H(r) = \frac{n-1}{r}\sqrt{V}.$$

The second fundamental form is $A(e_i, e_j) = \frac{1}{r}\sqrt{V}\delta_{ij}$, so $|A|^2 = (n-1)\frac{V}{r^2}$ and $|A|^2 - H^2 = -(n-1)(n-2)\frac{V}{r^2}$. The scalar curvature of the induced metric on S_r (which is the standard metric) is $R^\Sigma = \frac{(n-1)(n-2)}{r^2}$. The second variation formula:

$$\sqrt{V}\partial_r(\sqrt{V}\partial_r)|S_r| = -\omega_{n-1}r^{n-1}[Ric(\nu,\nu) + |A|^2 - H^2],$$

quickly leads to:

$$\frac{n-1}{2r}V' + \frac{(n-1)(n-2)}{r^2}V = -[Ric(\nu,\nu) + |A|^2 - H^2],$$

and combining with the above:

$$Ric(\nu,\nu) = -\frac{n-1}{2r}V'.$$

Now the scalar curvature of $M = (R^n, g)$ may be obtained from the twice-traced Gauss formula:

$$R^M = R^\Sigma + 2Ric(\nu,\nu) + |A|^2 - H^2 = \frac{(n-1)(n-2)}{r^2} - \frac{n-1}{r}V' - (n-1)(n-2)\frac{V}{r^2},$$

$$R^M = \frac{n-1}{r^2}[(n-2)(1-V) - rV'], \text{ or } R^M - 2Ric(\nu,\nu) = \frac{(n-1)(n-2)}{r^2}(1-V).$$

Thus the differential equation giving metrics of constant scalar curvature $R^M \equiv \kappa n(n-1)$ is:

$$r^2 n\kappa = (n-2)W + rW', \quad W = 1 - V,$$

which is easily solved, with general solution:

$$W = \kappa r^2 + \frac{2m}{r^{n-2}},$$

or

$$V = 1 - \frac{2m}{r^{n-2}} - \kappa r^2,$$

which is Schwarzschild if $\kappa = 0$, and Kottler (Anti-de Sitter-Schwarzschild/de Sitter-Schwarzschild) if $\kappa \neq 0$.

Application: the PMT for spherically symmetric metrics.

For a metric of the above form, we have $e = g - \delta = (\frac{1}{V} - 1)dr^2$, and hence $e_{ij} = (\frac{1}{V} - 1)\frac{x^i x^j}{r^2}$ and:

$$\partial_k e_{ij} = (\frac{1}{V} - 1)\frac{\delta_{ik}x^j + \delta_{jk}x^i}{r^2} - [(\frac{V'}{V^2} + \frac{2}{r^2}(\frac{1}{V} - 1)]\frac{x^i x^j x^k}{r^2}.$$

For the mass one-form, this gives (with implied summation over i):

$$\partial_i r_{ij} - \partial_j e_{ii} = (n-1)(\frac{1}{V} - 1)\frac{x^j}{r^2}.$$

And for the mass integral over S_r:

$$\oint_{S_r} (\partial_i r_{ij} - \partial_j e_{ii})\frac{x^j}{r} d\sigma_r^0 = (n-1)\omega_{n-1} r^{n-2}\frac{1-V}{V}.$$

The asymptotic decay of g to δ takes the form: $V = 1 + O_2(r^{-q}$, and this implies $\frac{1-V}{V} - (1-V) = (1-V)(\frac{1}{V} - 1) = O(r^{-2q})$. Since $q > (n-2)/2$ is assumed, this difference (times r^{n-2}) vanishes in the limit, and we may write:

$$m(g) = \lim_{r\to\infty} \frac{1}{2} r^{n-2}(1-V).$$

On the other hand, from the above: $R^g \geq 0 \Leftrightarrow (n-2)(1-V) \geq rV'$, or $r^{n-2}(1-V)$ is nondecrasing. Since the metric is defined in all of R^n:

$$0 = \lim_{r\to 0_+} \frac{1}{2} r^{n-2}(1-V) \leq \lim_{r\to\infty} \frac{1}{2} r^{n-2}(1-V) = m(g).$$

Additionally, if equality holds we have: $\frac{1}{2} r^{n-2}(1-V) \equiv m$, a constant; equivalently: $V = 1 + \frac{2m}{r^{n-2}}$, so g is the spatial Schwarzschild metric with parameter m (defined for $r > (2m)^{1/n-2}$.)