ASCOLI-ARZELA THEOREM-notes

If (X, d) is a metric space, $(E, ||\cdot||)$ a Banach space, the space $C_E^b(X)$ of bounded continuous functions from X to E (with the supremum norm) is a Banach space, usually infinite dimensional. Thus we don't expect arbitrary bounded sets in $C_E^b(X)$ to have compact closure.

It is very useful to have a criterion that guarantees a sequence in $C_E^b(X)$ has a convergent subsequence (meaning, uniformly convergent in X). Although 'bounded' is not enough, it turns out that a necessary and sufficient criterion exists.

Definitions. In the following definitions, we emphasize *sequential compactness*. This is equivalent to compactness (covering definition) for second-countable spaces, a hypothesis that is satisfied in the cases we consider.

(i) A subset $A \subset E$ of a Banach space is *precompact* if its closure $\bar{A}$ is compact; equivalently, if any sequence (v_n) in A has a convergent subsequence. (The limit may fail to be in A.)

A family $\mathcal{F} \subset C_E^b(X)$ is *precompact* if any sequence $(f_n)_{n\geq 1}$ of functions in $\mathcal{F}$ has a convergent subsequence (that is, a subsequence converging uniformly in X to a function $f \in C_E^b(X)$, not necessarily in $\mathcal{F}$).

(ii) Given a family $\mathcal{F} \subset C_E^b(X)$ and $x \in X$, we set:

$$\mathcal{F}(x) = \{v \in E; v = f(x) \text{ for some } f \in \mathcal{F}\} \subset E; \quad \mathcal{F}(X) = \bigcup_{x\in X} \mathcal{F}(x).$$

(iii)$\mathcal{F} \subset C_E^b(X)$ is *equicontinuous* at $x_0 \in X$ if for all $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, x_0) > 0$ so that

$$(\forall x \in X)[d(x, x_0) < \delta \Rightarrow (\forall f \in \mathcal{F})||f(x) - f(x_0)|| < \epsilon].$$

(The point, of course, is that the same δ works for all $f \in \mathcal{F}$.)

(iv) $\mathcal{F} \subset C_E^b(X)$ is *uniformly equicontinuous* on a subset $A \subset X$ if for all $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ so that

$$(\forall x, y \in A)[d(x, y) < \delta \Rightarrow (\forall f \in \mathcal{F})||f(x) - f(y)|| < \epsilon].$$

Exercise 1. If (X, d) is a compact metric space, any family $\mathcal{F} \subset C_E^b(X)$ which is equicontinuous at each $x \in X$ is, in fact, uniformly equicontinuous on X.

Arzela-Ascoli theorem. Let (X, d) be a *compact* metric space. Then $\mathcal{F} \subset C_E(X)$ is precompact provided $\mathcal{F}$ satisfies:
(i) $\mathcal{F}(x)$ is precompact in E, for each $x \in X$ and
(ii)$\mathcal{F}$ is equicontinuous at each $x \in X$.

Conversely, if $\mathcal{F}$ is precompact, then it is uniformly equicontinuous on X, and $\mathcal{F}(X)$ is precompact in E.

That the conditions (i) and (ii) are natural follows from the following exercises:

Exercise 2. Let (X, d) be a metric space. If a sequence $(f_n)_{n\geq 1}$ of functions in $C_E(X)$ converges to $f \in C_E(X)$ uniformly on X, then the family $\mathcal{F} = \{f_1, f_2, \ldots, f_n, \ldots f\}$ is equicontinuous at each $x_0 \in X$. (*Hint:* 3ϵ argument, using that f is continuous at x_0.)

Example. The sequence $f_n(x) = \frac{\sin(nx)}{n}(x \in R, n \geq 1)$ converges to zero uniformly on R, hence is equicontinuous at each $x \in R$.

Example. The sequence $f_n(x) = \frac{x^2}{n}$ $(n \geq 1, x \in R)$ is uniformly equicontinuous in each compact interval $[-M, M] \subset R$. (This is seen directly from the definition.)

Exercise 3. If (f_n) is a sequence in $C^b_E(X)$ and $f_n \to f$ uniformly on X, then the same family $\mathcal{F}$ as in Exercise 2 satisfies: $\mathcal{F}(X)$ is a precompact subset of E.

Case of R^n. If E is a finite-dimensional Banach space, a subset $A \subset E$ is precompact if and only if it is bounded (Bolzano-Weierstrass). We have:

Corollary: Ascoli-Arzela in R^n: Let (X, d) be a *compact* metric space. Then $\mathcal{F} \subset C(X; R^n)$ is precompact provided $\mathcal{F}$ satisfies:
(i) $\mathcal{F}(x)$ is a bounded subset of R^n, for each $x \in X$ ('$\mathcal{F}$ is equibounded') and
(ii)$\mathcal{F}$ is equicontinuous at each $x \in X$.

Conversely, if $\mathcal{F}$ is precompact, then it is uniformly equicontinuous on X, and $\mathcal{F}(X)$ is bounded in R^n.

Main example of equicontinuity. A family $\mathcal{F} \subset C^1(U, F)$ of C^1 functions from an open *convex* set $U \subset E$ of a Banach space to a Banach space F (think of $E = R^n$, or R, and $F = R^m$ if you want) is automatically equicontinuous (uniformly on each compact set $K \subset U$), provided we have a bound of the type:

$$||f'(x)|| \leq M_K \text{ for } x \in K, \text{ for each } K \subset U \text{ compact },$$

where the same M_K works for all $f \in \mathcal{F}$. (Note we always have this bound for a constant M depending on K and on f, by continuity of the differential):

$$f' : U \to \mathcal{L}(E, F)$$

The reason is the *Mean Value Inequality*: for $x, y \in K$ we have:

$$||f(x) - f(y)|| \leq \sup\{||f'(z)||; z \in K\}\|x - y\| \leq M_K||x - y||,$$

where $K \subset U$ is any compact subset containing the line segment from x to y. If this bound on f' holds, and in in the finite-dimensional case, the condition '$\mathcal{F}(x)$ is bounded, for each $x \in U$', only needs to be checked at one point $x_0 \in U$. This follows directly from

$$||f(x)|| \leq ||f(x_0)|| + M_K||x - x_0||$$

(with $K \subset U$ compact, containing the line segment from x_0 to x.)

Derivative bounds of this kind often arise in the context of solutions to differential equations.

Exercise 4. The sequence of functions $f_n(x) = nx^2$ has bounded derivatives at the point 0 but is not equicontinuous at 0 (prove this). Why does this not contradict the above discussion?

The proof of (the main direction of) the Ascoli-Arzela theorem follows three steps:

1) Equicontinuity + pointwise convergence on a dense subset $\Rightarrow$ uniform convergence on any compact set.

2) Precompactness of $\{f_n(d)\}_{n \geq 1}$ in E for each d in a countable set D $\Rightarrow$ pointwise convergence on D for a subsequence.

3) Compact metric spaces are separable (that is, one may find a countable dense subset.) (**Exercise 5.**)

Proposition 1. Let (X, d) a metric space, $D \subset X$ a dense subset, $(f_n)_{n \geq 1}$ a sequence in $C_E(X)$, equicontinuous at each $x \in X$. Then if f_n converges pointwise at each $d \in D$, then in fact (f_n) converges uniformly in each compact subset $K \subset X$.

Proof. Let $\epsilon > 0$ and $K \subset X$ compact be given. We need to show $|f_n(x) - f_m(x)|$ is small for all $x \in K$, if $m, n \geq N$, where $N = N(\epsilon)$.

First, for each $d \in D$: $|f_n(d) - f_m(d)| < \epsilon$ for $m, n \geq N(d)$.

By equicontinuity, for each $x \in X$ we may find an open ball $B_\delta(x) \subset X$ so that $|f_n(y) - f_n(x)| < \epsilon$ for all $n \geq 1$, and all $y \in B_\delta(x)$, $\delta = \delta(x)$. Taking a finite subcover of the covering of K by these balls, we have $K \subset \cup_{i=1}^{M} B_{\delta_i}(x_i)$. By density, we may find, for each $i = 1, \dots M$, a point $d_i \in B_{\delta_i}(x_i) \cap D$.

If $x \in K$, choosing an i so that $x \in B_{\delta_i}(x_i)$, considering the corresponding $d_i \in D \cap B_{\delta_{i(x_i)}}$ and letting $N = \max\{N(d_i), 1 \le i \le M\}$, we have, for $m, n \ge N$:

$$|f_n(x) - f_m(x)| \le |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_n(d_i)| + |f_n(d_i) - f_m(d_i)|$$

$$+|f_m(d_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| < 5\epsilon,$$

so (f_n) is Cauchy uniformly on K, as desired.

Remark: Lebesgue number of a covering. This concept is often useful in proofs involving compact sets. A number $r > 0$ is a *Lebesgue number* of an open covering $X \subset \cup_{\lambda \in \Lambda} U_\lambda$ (of a metric space X) if any two points of X that are r-close ($d(x, y) \le r$) are in the same U_λ.

Not every open covering (even a finite one) has a Lebesgue number (example: the open covering of $R \setminus \{0\}$ by $U_1 = \{x > 0\}, U_2 = \{x < 0\}$).

Any open covering $K \subset \cup_{\lambda \in \Lambda} U_\lambda$ of a compact set K has a Lebesgue number. If not, it would be possible to find $x_n, y_n \in K$ with $d(x_n, y_N) \le 1/n$, but (for any $n \ge 1$ no U_λ containing both x_n and y_n. Passing to a subsequence we have $x_{n_j} \to x_0 \in K$, and hence $y_{n_j} \to x_0$. Let U_λ be a set in the covering containing x_0. Then x_{n_j} and y_{n_j} are both in U_λ for j sufficiently large, contradiction.

Proposition 2. (Cantor-Tychonoff). Let D be a countable set. Any sequence of functions $f_n : D \to E$ such that the set $\{f_n(d); n \ge 1\}$ is precompact for each $d \in D$ has a subsequence which is pointwise convergent in D.

Proof. Standard 'diagonal argument'. The details:
Let $D = \{d_1, d_2, \dots\}$. Then $(f_n(d_1))_{n \ge 1}$ is precompact in E, so there exists $N_1 \subset \mathbb{N}$ so that $(f_N(d_1))_{n \in N_1}$ converges to a point in E, which we call $f(d_1)$.

Since $(f_n(d_2))_{n \ge 1}$ is precompact in E, we may find $N_2 \subset N_1$ so that $(f_n(d_2))_{n \in D_2}$ converges in E, and we call the limit $f(d_2)$.

Proceeding in this fashion, we define $f(d_n)$ for each $n \ge 1$. Now define an infinite set $N_0 \subset \mathbb{N}$ as follows: for each $i \ge 1$, we let the $i^{th.}$ element of N_0 be the $i^{th.}$ element of the set N_i. We claim $(f_n(d_i))_{n \in N_0}$ converges to $f(d_i)$, for each $i \ge 1$.

Indeed it suffices to observe that, beginning with its i^{th} element, N_0 is a subset of N_i. Given $\epsilon > 0$, find $m_i > i$ so that $n \in N_i, N > m_i \Rightarrow |f_n(d_i) - f(d_i)| < \epsilon$. Then the same is true if $n > m_i$ and $n \in N_0$.

Proof of the Ascoli-Arzela theorem. Let $\mathcal{F} \subset C_E(X)$ satisfy conditions (i) and (ii) in the theorem. Let $D \subset X$ be a countable dense set (using the fact X is separable.). Then if $(f_n)_{n\geq 1}$ is a sequence in $\mathcal{F}$, since the set $\{f_n(d)\}_{n\geq 1}$ is precompact in E for each $d \in D$ (and D is countable), by Proposition 2 a subsequence of (f_n) converges pointwise in D to $f : D \to E$. Since D is dense in X, equicontinuity of $\mathcal{F}$ at each point of X and Proposition 1 imply we may extend f from D to a (continuous) function $f : X \to E$, so that the same subsequence of (f_n) converges to f, uniformly on X.

For the converse, suppose $\mathcal{F}$ is (sequentially) precompact in $C_E(X)$, where X is compact metric. If $\mathcal{F}$ is not uniformly equicontinuous on X, we may find an $\epsilon_0 > 0$ and, for each integer $n \geq 1$, a function $f_n \in \mathcal{F}$ and points $x_n, y_n \in X$ so that $d(x_n, y_n) \leq \frac{1}{n}$ but $|f_n(x_n) - f_n(y_n)| \geq \epsilon_0$. By pre compactness, a subsequence of (f_n) converges uniformly to $f \in C_E(X)$, uniformly in X. f is uniformly continuous on X (since X is compact), yet $d(x_n, y_n) \to 0$ while $|f(x_n) - f(y_n)| \geq \epsilon_0$, contradiction.

If $\mathcal{F}(X)$ is not precompact, we may find sequences (f_n) in $\mathcal{F}$ and (x_n) in X so that the sequence $f_n(x_n)$ in E has no convergent subsequence. But $\mathcal{F}$ is assumed precompact, so (f_n) has a subsequence converging uniformly to a continuous function f, while a further subsequence of (x_n) converges to $x_0 \in X$ (by compactness). This implies a subsequence of $(f_n(x_n))$ converges to $f(x_0)$, contradiction.

It is important for many applications to extend the theorem to the case where X is not compact.

Recall a metric space (or Hausdorff topological space) X is *locally compact* if each point has a relatively compact open neighborhood (that is, one with compact closure.)

Exercise. Let X be locally compact, $C \subset X$ a compact subset. Then we may find an open set V containing C, with compact closure $\bar{V}$.

Definition. A *locally compact* metric space (X, d) is *σ-compact* (pronounced 'sigma-compact') if it is the union of countably many compact subsets: $X = \cup_{i\geq 1} K_i$, with $K_i \subset X$ compact (and we may assume $K_i \subset K_{i+1}$).

Proposition. X (locally compact) is σ-compact if (and only if) X can be written as a countable union $X = \cup_{i=1}^{\infty} U_i$ of relatively compact open sets U_i, where $\bar{U}_i \subset U_{i+1}$ for each $i \geq 1$.

Proof. Only one direction requires proof. Assume $X = \cup_{n\geq 1} K_n$, with each K_n compact. From the exercise, there is a relatively compact open set U_1 containing K_1. Proceeding inductively, we choose U_{n+1} to be a relatively compact open set containing the compact set $\bar{U}_{n-1} \cap K_n$. The sets U_n clearly satisfy the claim. ([Dugundji's *Topology*, p.241])

Exercise. Let X be locally compact and σ-compact, and consider sets U_n as in the proposition. Show that any compact set $C \subset X$ is contained in some U_n. (*Hint:* consider the open covering of $\{U_n \cap C; n \geq 1\}$ of C.)

We say a sequence of functions $f_n : X \to E$ (X metric, E Banach) converges to $f : X \to E$ *uniformly on compact sets* if $(\forall \epsilon > 0)(\forall K \subset X$ compact), we may find an $N \geq 1$ (depending on ϵ and on K) so that:

$$n \geq N \Rightarrow \sup_{x \in K} ||f_n(x) - f(x)|| < \epsilon.$$

Theorem. Let X be a σ-compact metric space, $\mathcal{F} \subset C(X; E)$ a family of continuous functions (E Banach). If $\mathcal{F}$ is equicontinuous at each $x \in X$ and $\mathcal{F}(x)$ is precompact in E for each $x \in X$, then any sequence of functions in $\mathcal{F}$ has a subsequence converging uniformly on compact sets (to a function $f \in C(X; E)$).

Proof. Follows from the Ascoli-Arzela theorem and a diagonal argument (like the one used in Proposition 2.)

Exercise 6. Prove this theorem in detail, following the idea just given.

Exercise 7. Prove that any $\sigma-$ compact metric space is separable (i.e., contains a countable dense subset.)

An extension of Arzela-Ascoli.

Definition. Let $(f_n)_{n\geq 1}$ be a sequence of continuous functions in R^N. We say f_n converges "partially uniformly on compact sets" if there exists an open set $A \subset R^N$ and $f_0 : A \to R$ continuous, so that $f_n \to f_0$ uniformly on compact subsets of A and $|f_n| \to \infty$ uniformly on compact subsets of $R^N \setminus \bar{A}$. (Note that A may be empty.)

Proposition. Let $\mathcal{F} \subset C(R^n)$ be a family of functions, uniformly equicontinuous on compact subsets of R^N. Then any sequence of functions of $\mathcal{F}$ converges partially uniformly on compact sets.

Proof. Consider the uniformly continuous function $\phi : R \to (-1,1)$, $(2/\pi)\arctan x$. Let $\mathcal{G}$ be the family of continuous functions from R^N to $(-1,1)$: $\mathcal{G} = \{\phi \circ f; f \in \mathcal{F}\}$. Since any $g \in \mathcal{G}$ satisfies $|g| < 1$ on R^N, the hypotheses of the usual Arzela-Ascoli are satisfied. Hence if (f_n) is any sequence in $\mathcal{F}$, the sequence $g_n = \phi \circ f_n : R^N \to (-1,1)$ has a subsequence g_{n_j} converging uniformly on compact subsets of R^N to a continuous function $g_0 : R^N \to [-1,1]$.

Let $A = g_0^{-1}(-1,1)$ (open in R^N), $R^N \setminus A = g^{-1}(\{-1,1\})$ (closed in R^N). Then the sequence $\phi^{-1} \circ (g_{n_j})_{|A} = (f_{n_j})_{|A}$ converges uniformly on compact subsets of A to $f_0 := \phi^{-1} \circ (g_0)_{|A}$.

It is also easy to show that $|f_{n_j}| \to \infty$, uniformly on compact subsets of $R^N \setminus \bar{A}$.

Translation families. For $f \in UC(R^N)$ (uniformly continuous, not necessarily bounded) the translation family is $\mathcal{T}_f = \{f_t; t \in R^N\} \subset UC(R^N)$, $f_t(x) = f(x-t)$. Clearly $\mathcal{T}_f$ satisfies the hypothesis of the proposition, hence is sequentially precompact in the sense of partial uniform convergence on compact sets.

An example from the Calculus of Variations. Consider the variational problem (optimizing in a set of functions):

$$\text{minimize } \Phi[f] := \int_{-1}^{1} f(t)dt$$

over the set$\mathcal{F} = \{f : [-1,1] \to [0,1] \text{ continuous }, f(-1) = f(1) = 1\}$.

1) Considering the sequence $f_n(x) = x^{2n}$ in $[-1,1]$, we see that the infimum is 0 and it is not attained within this family (since the area under the graph is always positive).

2) If we add the condition that f is Lipschitz in $[-1,1]$ (with constant $c > 0$), the Ascoli-Arzela theorem implies any minimizing sequence has a uniformly convergent subsequence. Thus the infimum is achieved (for this family): a minimizer of Φ can be found in the class

$$\mathcal{F}_c = \{f \in \mathcal{F} | f \text{ is } c\text{-Lipschitz in}[-1,1]\}.$$

3) In fact it is easy to see that (under a c-Lipschitz condition) the infimum is attained by an even in x, piecewise-linear function. (Graphs drawn in class.)

Exercise 8. For each $c > 0$, let $f_c : [-1, 1] \to [0, 1]$ be the c-Lipschitz function:

$$f_c(x) = \max\{1 + c(|x| - 1), 0\}, \quad x \in [-1, 1].$$

(i) Sketch the graph of f_c, in the cases $c > 1$, $c = 1$, $c < 1$.
(ii) Prove that if $f(1) = 1$, $f(x) \geq 0$ in $[0, 1]$ and f is c-Lipschitz in $[0, 1]$, then $f \geq f_c$ in $[0, 1]$. (And similarly in $[-1, 0]$.)
(iii) Explain why this implies that, for any $f \in \mathcal{F}_c$, $\Phi[f] \geq \Phi[f_c]$. Thus f_c is a minimizer of Φ in $\mathcal{F}_c$. Compute the minimum value $\Phi[f_c]$.

Problems.

1. Show there does not exist a sequence of continuous functions $f_n : [0, 1] \to R$ converging pointwise to the function $f : [0, 1] \to R$ given by $f(x) = 0$ for x rational, $f(x) = 1$ for x irrational.

2. Given $f : R \to R$ an arbitrary function, consider the sequence of translates $f_n(x) = f(x + n)$, $n \geq 1$. Then f_n converges uniformly on $[0, \infty)$ to the constant function L if, and only if, $\lim_{x\to\infty} f(x) = L$.

3. If each $f_n : X \to E$ (X metric, E Banach) is uniformly continuous on X and $f_n \to f$ uniformly on X, then f is uniformly continuous on X.(X is a metric space.)

4. There is no sequence of polynomials converging either to $1/x$ or to $\sin(1/x)$ uniformly on the open interval $(0, 1)$.

5. Find a sequence of functions $f_n : [0, 1] \to R$ which converges uniformly on $(0, 1)$, but not on $[0, 1]$.

6. If $f_n \to f$ uniformly on X (where $f_n, f : X \to E$, X metric, E Banach) and $g_n \to g$ uniformly on E ($g_n, g : C \to V$, $C \subset E$, F, V Banach) where $f_n(X) \subset C, f(X) \subset C$ and g is uniformly continuous on X, then $g_n \circ f_n \to g \circ f$ uniformly on X. Do we need to assume anything about the f_n, f or g_n?

7. A monotone sequence of real-valued functions is uniformly convergent provided it has a subsequence with this property.

8. If a sequence of real-valued monotone functions (with domain R) converges pointwise to a continuous function on an interval $I \subset R$, then the convergence is uniform on each compact subset of I.

9. If $\lim f_n(c) = L$ exists (for some $c \in R$, where $f_n : I \to R$ and $I \subset R$ is an interval containing c) and the sequence of first derivatives (f_n')

converges to 0 uniformly on I, then $f_n \to L$ uniformly on each compact subset of I.*Example:* $f_n(x) = \sin(\frac{x}{n})$.

10. A sequence of polynomials of degree $\leq k$, uniformly bounded in a compact interval, is equicontinuous on this interval.

11. Let (f_n) be an equicontinuous and pointwise bounded sequence of functions in $C_E(X)$ (E Banach and finite-dimensional , X compact metric.) If every uniformly convergent subsequence has the same limit $f \in C_E(X)$, then f_n converges to f uniformly on X.