COMPACTNESS, COUNTABILITY, FUNCTION SPACES: EXAMPLES

Examples 1 to 4 consider metrizability for the pointwise and uniform topologies.

Ex 1. $X = \mathcal{F}_p(R, R)$ *is not first countable.*

Recall the notation for a local basis at $f \in X$ for the topology of pointwise convergence:

$$A_f(t_1, \dots t_k; \epsilon_1, \dots, \epsilon_k) = \{g \in X; |g(t_i) - f(t_i)| < \epsilon_i, i = 1, \dots, k).$$

Let $f \in X$, suppose we had $\{V_1, \dots, V_n, \dots\}$ countable basis at f. Each V_n would contain an element of the local basis at f for the pointwise topology:

$$A_n = A_f(t_{n1}, \dots, t_{nk}; \epsilon_{n1}, \dots, \epsilon_{nk}) \subset V_n.$$

Thus the A_n would also form a countable basis of neighborhoods of f.

The set of all t appearing in these sets is also countable, so there exists $t_0 \in R$ not occurring in any of them. Consider the basis set $A_0 = A_f(t_0, 1)$. For functions in A_n there is no restriction on the value at t_0, so certainly there is $g_n \in A_n$ with $|g_n(t_0) - f(t_0)| \geq 1$, so $g_n \notin A_0$, showing A_0 is not contained in any A_n. So the A_n can't be a local basis at f.

This argument also works to show the compact space $\mathcal{F}_p(R; [0, 1])$ is not first countable, hence not metrizable.

Ex 2. Again let $X = \mathcal{F}_p(R, R)$, and let $S \subset X$ be the set of characteristic functions of finite sets. *Claim:* The constant function $g \equiv 1 \in X$ is in the closure of S, but is not the pointwise limit of functions in S.

Indeed given any basic neighborhood $A = A_g(t_1, \dots, t_n, \epsilon)$, the characteristic function of the set $\{t_1, \dots, t_n\}$ is in $A \cap S$. Now if $f_n \in S$, $f_n \to g$ pointwise on R, the set of $t \in R$ such that $f_n(t) \neq 0$ for some n is countable, so there exists $t_0 \in R$ such that $f_n(t_0) = 0$ for all n. This contradicts $f_n(t_0) \to 1$.

Ex. 3. *The uniform topology in* $F_u = \mathcal{F}_u(X, Y)$ *is metrizable (*X*: set; (*Y, d*) metric space.)*

If d is a bounded metric (or if X is a finite set), we can just take the sup metric:

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

But if X is infinite and (Y, d) unbounded, the supremum is not necessarily finite, so we can't just take the *sup* metric. Recall a local basis for $\mathcal{F}_u$ at f is given by the sets:

$$B_f(\epsilon) = \{g \in \mathcal{F}_u; d(f(c), g(x)) < \epsilon, \forall x \in X\}.$$

Definition: two metrics d, d' in Y are *uniformly equivalent* if the identity is a uniform homeomorphism. (A homeomorphism f of metric spaces is *uniform* if both f and f^{-1} are uniformly continuous.) For example, the metrics $d(x, y)$, $\min\{d(x, y), 1\}$, $\frac{d(x,y)}{1+d(x,y)}$ are all uniformly equivalent (exercise.)

Proposition. If M, N are metric spaces and $\varphi : M \to N$ is uniformly continuous, the map induced by composition

$$\varphi_* : \mathcal{F}_u(X, M) \to \mathcal{F}_u(X, N), \quad \varphi_*(f) = \varphi \circ f$$

is continuous. This is very easy to prove (try it!) The converse holds if X is infinite.

As a corollary, if φ is a uniform homeomorphism, φ_* is a homeomorphism. If φ is bounded (for instance if d_N is bounded), then φ_* maps $\mathcal{F}_u(X, M)$ to $\mathcal{B}_u(X, N)$, the space of bounded maps, with the uniform topology (which is metrizable via the *sup* norm.)

Thus two uniformly equivalent metrics in Y define equivalent topologies in $\mathcal{F}_u(X, Y)$, and if one of them is bounded we have that $\mathcal{F}_u(X, Y)$ and $\mathcal{B}_u(X, Y)$ are homeomorphic, the latter space being explicitly metrizable (when Y is given a bounded metric).

Since any metric in Y is uniformly equivalent to a bounded one, we see that $\mathcal{F}_u(X, Y)$ is always metrizable.

The following example shows this argument utterly fails, without the assumption of uniform equivalence of the metrics.

Ex. 4. Let X be an infinite set. Then $\mathcal{F}_u(X, R)$ (where R has the usual metric $d(x, y) = |x - y|$) is disconnected, since the subspace $\mathcal{B}_u(X, R)$ (bounded functions) is open, closed, non-empty, and not the whole space. Let $h : R \to (-1, 1)$ be the homeomorphism $h(x) = x/(1+|x|)$, and define in R the bounded metric $d_1(x, y) = |h(x) - h(y)|$, which is equivalent to d, but not uniformly. (The identity $(R, d_1) \to (R, d)$ is not uniformly continuous.)

Let $R_1 = (R, d_1)$. Then $\mathcal{F}_u(X, R_1)$ is metrizable (since d_1 is bounded), and in fact $h_* : \mathcal{F}_u(X, R_1) \to \mathcal{F}_u(X, (-1, 1))$ is an isometry (between d_1 and d), in particular a homeomorphism. Now $\mathcal{F}_u(X, (-1, 1))$ is a convex subset of the normed vector space $\mathcal{B}_u(X,R)$, in particular a connected space. Hence $\mathcal{F}_u(X,R_1)$ is connected as well, and so cannot be homeomorphic to $\mathcal{F}_u(X,R)$.

This example should be carefully compared with the argument in Example 3.

Examples 5, 6 review the topic 'compactness v. sequential compactness'.

Ex. 5 *Compactness vs. sequential compactness.* Consider the conditions (for a given space X). *Note:* sequences always assumed injective.

(1) Every sequence in X has a cluster point (x is a cluster point of (x_n) if any open neighborhood of x contains infinitely many x_n.)

(2) Every sequence in X has a subsequence converging to a point of X (sequential compactness.)

(3) X is compact.

Then:

a) (3)$\Rightarrow$ (1), for any space:

Proof: If (x_n) has no cluster point, for all x there is an open neighborhood V_x containing only finitely many sequence elements. Taking a finite subcover, we see this would imply the set $\{x_n; n \geq 1\}$ is finite, contradiction.

b) (2)$\Rightarrow$ (1), for any space (clear);

c) If X is first-countable, (1)$\Rightarrow$ (2). In particular, for 1st countable spaces, compact implies sequentially compact.

Proof: Let x be a cluster point of (x_n), $(V_j)_{j\geq 1}$ a countable basis of neighborhoods of x, where we may assume $V_{j+1} \subset V_j$. Form a subsequence in the following way: take n_1 so that $x_{n_1} \in V_1$, then $n_2 > n_1$ so that $x_{n_2} \in V_2$,..., $n_{j+1} > n_j$ so that $x_{n_{j+1}} \in V_{j+1}$, etc. Then $x_{n_j} \to x$ as $j \to \infty$.

Note that for the compact space $\mathcal{F}_p(R,[0,1]))$, the characteristic function χ_Q of the rationals is a cluster point of the set of continuous functions (see Example 6 below), but there is no sequence of continuous functions converging pointwise to g.

d) If X is second countable, all are equivalent. (I.e. (1) or (2) imply (3), in particular compact and sequentially compact are equivalent for 2nd countable spaces (for example, for separable metric spaces).

Proof: Let $\mathcal{C}$ be an open cover of X. Second countable implies Lindelöf [Munkres p. 190-191], so there exists a countable subcover $\{U_i\}_{i\geq 1}$. Proceeding by contradiction, assume there is no finite subcover. Then take $x_1 \in U_1, x_2 \in U_1 \setminus U_2, x_3 \in U_3 \setminus (U_1 \cup U_2), ...x_n \in U_n \setminus (U_1 \cup \ldots \cup U_{n-1})$,

...This defines an infinite sequence (x_n) so that $x_n \notin U_m$ if $n > m$. Let x be a cluster point of the sequence. Then $x \in U_i$ for some i (since the $\{U_i\}$ cover X), and then $x_n \in U_i$ for some $n > i$, contradiction.

Ex. 6. Consider again the compact space $X = \mathcal{F}_p(R, [0,1])$. Let $\{r_n\}_{n\geq 1}$ be an enumeration of the rationals. For each $E_j = \{r_1, \ldots, r_j\}$, let $f_j \in X$ be the characteristic function of E_j. Then f_j converges pointwise to the characteristic function $\chi_Q \in X$ of the rationals, and on the other hand each f_j is the pointwise limit of a sequence of continuous functions in X. So if $S \subset X$ is the set of continuous functions, χ_Q is a cluster point of X: any neighborhood of χ_Q contains infinitely many points of S (check this.) However, χ_Q cannot be the pointwise limit of any sequence in S, since it is discontinuous everywhere. (Recall the set of continuity of a pointwise limit of continuous functions is 'residual'.)

Examples 7, 8, 9 refer to compactly generated spaces.

Ex. 7. *First countable spaces X are compactly generated.*

Let $F \subset X$ such that $F \cap C$ is closed in C, for each $C \subset X$ compact. To show F is sequentially closed in X, let x_n be a sequence in F, and assume $\lim x_n = x$ in X. Let $C = \{x_n; n \geq 1\} \cup \{x\}$. Then C is compact: if $\mathcal{U}$ is an open cover of C, we may, for each n, pick $U_n \in \mathcal{U}$ so that $x_n \in U_n$; and also $U_0 \in \mathcal{U}$ containing x. Now let N be such that $x_n \in U_0$ for $n \geq N$. Then $U_0, U_1, \ldots U_n$ is a finite subcover of C.

Thus $F \cap C$ is (sequentially) closed in C (first countability is inherited by subspaces.) Since $x_n \in F \cap C$, also $x \in F \cap C$, proving $x \in F$, so F is sequentially closed.

Ex 8. *Locally compact spaces X are compactly generated.*

Let $A \subset X$ be such that $A \cap C$ is open in C, for all $C \subset X$ compact. Given $x \in A$, we want to find W open in X and containing x, so that $W \subset A$.

Since X is locally compact, there exist V open in X and C compact in X, so that $x \in V \subset C$. Then $A \cap C$ is open in C (in the subspace topology), while $x \in A \cap C$. So there exists U open in X and containing x so that $U \cap C \subset A \cap C$. Now let $W = U \cap V$, clearly open in X and containing x. $W \subset U$ and $W \subset V \subset C$, so $W \subset U \cap C \subset A$.

Ex. 9. *Example of a non-compactly generated space* [Willard 43H, p. 289]

$X = \mathcal{F}_p(R, R)$ (pointwise convergence) is not compactly generated. To see this consider the set:

$$T = \{f \in X; (\exists n \geq 1)(\exists F \subset R)\text{card } (F) \leq n, f(x) = 0 \text{ on } F, f(x) = n \text{ on } R\backslash F\}.$$

(That is, T is the set of functions equal to zero on a finite set, and equal to an upper bound for the cardinality of that set elsewhere.)

Then T is not closed (the identically 0 function is in the closure of T, but not in T.) But $T \cap C$ is compact (in particular closed in C), for all $C \subset X$ compact. (Exercise). *Outline:* Note that if C is compact, for each $x \in R$ there exists an $M_x > 0$ so that $|f(x)| \leq M_x$, for all $f \in C$. The definition of T then implies that, for each $x \in R$, the image $\{f(x); f \in T \cap C\}$ is a finite, hence compact subset of R. By Tychonoff's theorem, $T \cap C$ is compact.

The next examples 10, 11, 12 consider separability of certain function spaces with compact domain (with uniform topology, given by a metric or a norm.)

Ex. 10. The theorem in this example is of great importance in applications. For example, it implies the Banach space $C^1(K, R^n)$ (C^1 functions with values in R^n, defined on a compact convex $K \subset R^m$, with the C^1 sup norm) is separable.

Theorem. Let K be a compact metric space, (M, d) a separable metric space (in particular, with countable basis). The metric space $X = C_u(K, M)$ (continuous functions, uniform topology, *sup* metric) has countable basis.

Preliminary remark: If $L \subset K$ is compact and $f(L) \subset U$, where $U \subset M$ is open, then if $\epsilon = dist(f(L), M \setminus U)$ (a positive number, since $f(L)$ is compact) we have $d(f, g) < \epsilon \Rightarrow f(L) \subset U$. Indeed, if $x \in L$ and $y \in M \setminus U$:

$$d(g(x), y) \geq d(f(x), y) - d(f(x), g(x)) \geq d(f(L), M \setminus U) - d(f, g) > \epsilon - \epsilon = 0,$$

so necessarily $g(x) \neq y$, showing $g(x) \in U$.

Proof of theorem. Let $\mathcal{B}$ be a countable basis for M.

For each $n \geq 1$, fix once and for all a decomposition $K = K_1^n \cup \ldots \cup K_p^n$, where $diam(K_i^n) < 1/n$ and $p = p(n)$. (This is possible since K is 'totally bounded'). For each $n \geq 1$ and each p-tuple $\sigma = (B_1, \ldots, B_p)$ of elements of $\mathcal{B}$ (with the same cardinality $p = p(n)$), let

$$A(n, \sigma) = \{f \in C(K; M); f(K_i^n) \subset B_i, i = 1, \ldots, p\}.$$

Clearly the collection of all such $A(n, \sigma)$ (for varying n and σ) is countable. (Since, for each $n \geq 1$, there are only countably many $p(n)$-tuples of sets drawn from the countable collection $\mathcal{B}$). We claim this collection is a basis of open sets for X.

The $A(n,\sigma)$ are open: if $f \in A(n,\sigma)$, $f(K_i^n)$ is a compact subset of B_i, so $d(f(K_i^n), M \setminus B_i) = \epsilon_i > 0$, and if $\epsilon = \min_i \epsilon_i$ and $d(f,g) < \epsilon$, then also $g(K_i^n) \subset B_i$ for all $i = 1, \ldots, p$ (by the preliminary remark), so $g \in A(n,\sigma)$.

To prove $\mathcal{A}$ is a basis, it suffices to show that for each $f \in X$ and $\epsilon > 0$, we may find $A(n,\sigma)$ so that $f \in A(n,\sigma)$ and $A(n,\sigma) \subset B_f(\epsilon)$. Given the former, for the latter it suffices to show $diam(A(n,\sigma)) < \epsilon$. We know the compact set $f(K) \subset M$ is contained in the union of a finite number of sets of $\mathcal{B}$, each with diameter $< \epsilon$.

(Why? For each $y \in M$, the open ball of center y and radius $\epsilon/3$ is a union of sets from $\mathcal{B}$, all necessarily with diameter less than ϵ, and one of which contains y. Thus M is covered by open sets from $\mathcal{B}$ with diameter less than ϵ.)

Let η be a Lebesgue number of this finite open cover of $f(K)$.

By uniform continuity of f, there exists an $n \geq 1$ so that in the decomposition $K = K_i^n \cup \ldots \cup K_p^n$ we have $diam(f(K_i^n)) < \eta$ for all $i = 1, \ldots, p$, and hence each $f(K_i^n)$ is contained in a single set of this open cover of $f(K)$, giving sets $B_1, \ldots, B_p$ in $\mathcal{B}$ so that $f(K_i^n) \subset B_i$ (and $diam(B_i) < \epsilon$). This defines a p-tuple σ of sets in $\mathcal{B}$ so $f \in A(n,\sigma)$.

In addition, if $g, h \in A(n,\sigma)$ and $x \in K$, then $x \in K_i^n$ for some i and both g and h map K_i^n to B_i, so $d(g(x), h(x)) < \epsilon$ (since $diam(B_i) < \epsilon$). This shows $diam(A(n,\sigma)) < \epsilon$, as desired. This concludes the proof.

Example: In particular, the metric space $C(K)$ of continuous R-valued functions in K has a countable basis, and hence is separable.

Ex. 11. (*Example of an inseparable metric space of continuous functions.*) If, instead, we exchange M and K and consider $X = C(M,K)$ (M separable metric, K compact metric), then X (with the *sup* metric, or uniform convergence) is *not* necessarily separable. For an example, consider $X = C_u(R, [0,1])$ (with the *sup* metric).

The set $\mathcal{P}(\mathbb{Z})$ of subsets of $\mathbb{Z}$ is uncountable. To each $S \subset \mathbb{Z}$, associate a piecewise linear function f_S which is 1 on S, 0 on $\mathbb{Z} \setminus S$. Then if $S_1 \neq S_2$ are two different subsets of Z, $d(f_{S_1}, f_{S_2}) = 1$. So the set of all f_S is an uncountable, discrete subset of the metric space X, and therefore X cannot be separable (exercise.) In particular, it follows that, unlike $\mathcal{F}_p(R, [0,1])$ (pointwise convergence), $\mathcal{F}_u(R, [0,1])$ is *not compact.* (Since compact metric spaces are always separable–proof?)

Note that this example also shows the Banach space $C_u^b(R,R)$ (continuous bounded functions, with the sup norm) is *not separable.* Another example (with a compact domain!) is given next.

Ex.12. For fixed $0 < \alpha < 1$, consider the space $E_\alpha = C^\alpha[0,1]$ of real-valued Hölder-continuous functions on the unit interval, with the norm:

$$||f|| = |f(0)| + \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\alpha}.$$

It is easy to show this is a Banach space, and we claim it is not separable. The argument is the same as in the preceding example: we find an uncountable family $\{f_t\}_{t\in(0,1)}$ in E_α, with pairwise unit distance from each other. Namely, consider:

$$f_t(x) = 0, 0 \leq x \leq t; \quad f_t(x) = (x-t)^\alpha, t \leq x \leq 1.$$

We have: (i) $f_t \in E_\alpha$, with $||f_t|| = 1$. (ii)If $0 < t < t' < 1$, $||f_{t'} - f_t|| \geq 1$. Thus E_α is not separable.

To see this, note that a simple computation gives:

$$||f_{t'} - f_t|| = \sup_{x_1 \neq x_2} \frac{|(f_{t'} - f_t)(x_1) - (f_{t'} - f_t)(x_2)|}{|x_1 - x_2|^\alpha} \geq \frac{|(f_{t'} - f_t)(t) - (f_{t'} - f_t)(t')|}{|t' - t|^\alpha} = \frac{(t'-t)^\alpha}{(t'-t)^\alpha} = 1.$$

Remarks: (i) Spaces of Hölder-continuous functions are ubiquitous in PDE (where they are needed for sharp regularity results), usually with a different norm:

$$||f||_{C^\alpha} = ||f||_{sup} + \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\alpha}.$$

It is still true the space is inseparable for this norm: since it dominates the earlier norm, a countable dense set in this norm would also be dense for the earlier one.

(ii) Exactly the same argument, in the case $\alpha = 1$, shows that the space of Lipschitz functions in $[0,1]$ (with the usual norm, which makes it a Banach space) is also not separable. (Exercise.)

In the next examples we consider the questions of metrizability and countable basis for the spaces $\mathcal{F}_p(X,M), \mathcal{C}_u(X,M), \mathcal{C}_c(X,M)$ (X: top. space, (M,d): metric space.

For the pointwise topology, we just use known facts about the product topology to conclude:

For M metric (and with more than one point!), $\mathcal{F}_p(X, M)$ is metrizable if and only if X is countable (and then there is a natural metric.) And $\mathcal{F}_p(X, M)$ is second-countable if, and only if, X is countable and M is separable.

For the uniform topology: $\mathcal{C}_u(X, M)$ is always metrizable by the *sup* metric, replacing d by a uniformly equivalent bounded metric, if needed, as discussed in Example 3. (And this metric will be complete, if the metric on M is.) And we saw in Example 10 that $\mathcal{C}_u(X, M)$ is second countable (equivalently: separable), if X is compact metric and M is separable metric. On the other hand, Example 11 shows this fails if X is not compact.

Turning to the u.o.c topology, we focus on the case: X locally compact and σ-compact. Then there exists a compact exhaustion:

$$X = \bigcup_{i \geq 1} K_i; \quad K_i \text{ compact }, \quad K_i \subset int(K_{i+1}).$$

Examples 13, 14 deal with metrizability and separability for the topology of uniform convergence on compact sets.

Ex. 13. If X is locally compact, σ-compact (for instance, a topological manifold), then $\mathcal{C}_c(X, M)$ is second-countable if M is separable metric.

For each $i \geq 1$, let $\mathcal{B}_i$ be a countable basis for $\mathcal{C}_u(K_i, M)$ (using Example 10.) Then $\mathcal{B} = \bigcup_{i \geq 1} \mathcal{B}_i$ is a countable basis for $\mathcal{C}_c(X, M)$. To see this, consider a basic open neighborhood $B_f(K, \epsilon)$ of $f \in \mathcal{C}_c(X, M)$, and pick some $i \geq 1$ so that $K \subset K_i$. (Note: we need local compactness for this.) Let $g_i = f_{|K_i}$. Then:

$$B_f(K, \epsilon) \supset B_{g_i}(K_i, \epsilon) = \{h \in C(K_i, M); d(h(x), g_i(x)) < \epsilon, x \in K_i\}.$$

And this latter set (a basis element for the uniform topology over K_i) is the union of (countably many) open sets in $\mathcal{B}_i$.

Ex. 14. If X is locally compact, σ-compact, (M, d) metric, then $\mathcal{C}_c(X, M)$ is metrizable with a natural metric (complete, if the metric on M is.)

Indeed, we may take on $\mathcal{C}_c(X, M)$ the metric:

$$d(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{x \in K_i} \frac{d(f(x), g(x))}{1 + d(f(x), g(x))}.$$

The last two examples answer questions raised in class:

Ex. 15 *A subset of the real line which is both residual and a nullset.*

Let $\{r_1, r_2, \ldots\}$ be an enumeration of the rationals, and for each $n \geq 1$ and $j \geq 1$ let I_{nj} be an open interval with center r_n, length $1/2^{n+j}$. Then $A_j = \bigcup_{n \geq 1} I_{nj}$ is open and dense in R, hence $A = \bigcap_{j \geq 1} A_j$ is residual in R. But also $A \subset \bigcup_{n,j} I_{nj}$, and $\sum_{n,j} length(I_{nj}) = \frac{1}{2^j}$ can be made arbitrarily small, taking j large enough.

Corollary: Any subset of R is the union of a nullset and a set of first category. (Ref: J.C. Oxtoby, *Measure and Category*, Springer-Verlag 1980, p.5.)

Ex. 16. There exists a function f, differentiable in [0,1], whose derivative does not have constant sign on any non-degenerate interval in [0,1]. (Ref: *Twelve Landmarks of Twentieth Century Analysis* by Choimet and Queffélec, p.116.)