MATH 562, SPRING 2026–PROBLEM SET 1

1. Prove that if a metric space (M, d) contains an uncountable discrete set D (that is, for each $d \in D$, some open ball with center d intersects D only at d), then M cannot be second-countable.

2. Consider the space $X = C^1([a, b])$ of continuously differentiable functions on a compact interval $[a, b]$, with the norm:

$$||f|| = sup_{x\in[a,b]}(|f(x)| + |f'(x)|).$$

(i) Prove that X is a Banach space.

(ii) Prove that X is second-countable.

3. If (X, d) is a compact metric space, E a Banach space, any family $\mathcal{F} \subset C_E(X)$ which is equicontinuous at each $x \in X$ is, in fact, uniformly equicontinuous on X.

4. Let (X, d) be a metric space, E a Banach space. If a sequence $(f_n)_{n\geq 1}$ of functions in $C_E(X)$ converges to $f \in C_E(X)$ uniformly on X, then the family $\mathcal{F} = \{f_1, f_2, \ldots, f_n, \ldots f\}$ is equicontinuous at each $x_0 \in X$. (*Hint:* 3ϵ argument, using that f is continuous at x_0.)

5. (i) Prove that a compact metric space (M, d) is separable. (That is, there exists $D \subset M$ countable and dense.)

(ii) Prove that any $\sigma-$ compact metric space is separable (i.e., contains a countable dense subset.)

6. If each $f_n : X \to E$ (X metric, E Banach) is uniformly continuous on X and $f_n \to f$ uniformly on X, then f is uniformly continuous on X.

7. There is no sequence of polynomials converging either to $1/x$ or to $\sin(1/x)$ uniformly on the open interval $(0, 1)$.

8. Find (with proofs) a sequence of continuous functions $f_n : [0, 1] \to R$ which converges uniformly on $(0, 1)$, but not on $[0, 1]$.

9. If $\lim f_n(c) = L$ exists (for some $c \in R$, where $f_n : I \to R$ are continuously differentiable and $I \subset R$ is an interval containing c) and the sequence of first derivatives (f'_n) converges to 0 uniformly on I, then $f_n \to L$ uniformly on each compact subset of I. *Example:* $f_n(x) = \sin(\frac{x}{n})$.

10. Prove: A sequence of polynomials of degree $\leq k$, uniformly bounded in a compact interval, is equicontinuous on this interval.

11. Exercise 8, p.8 of the Ascoli-Arzela notes. (It is enough to do the case $c > 1$.)