MATH 562, SPRING 2026–PROBLEM SET 2

1. X is locally compact, σ-compact. $f_n : X \to \mathbb{R}^k$. If (f_n) is equicontinuous on compact sets and bounded at each point, there exists a subsequence converging u.o.c. to a continuous function. (You may use the A-A theorem for *compact* metric domains.)

2. Let $X = [0, \infty)$. Show that for each continuous $f : X \to \mathbb{R}$ there exists a sequence of the form:

$$p_k(x) = \sum_{n=0}^{n_k} a_n e^{-nx}$$

such that $p_k \to f$ uniformly on compact sets.

3. (i) If $f \in C(\mathbb{R}^n)$, there exists a sequence p_j of polynomials in n variables so that $p_j \to f$ u.o.c. in $\mathbb{R}^n$.

(ii) If $f(0) = 0$, we may require the approximating polynomials to satisfy $p_j(0) = 0$ for all j.

4. The set of continuous, piecewise linear functions is dense in $C(\mathbb{R})$ (for the u.o.c. topology). *Hint:* use the version of Stone-Weierstrass for lattices of continuous functions.

5. Prove: A complete metric space without isolated points is uncountable. (*Hint:* Baire property, complements of one-point sets.)

6. *Definition:* A subset $S \subset X$ of a topological space X is a G_δ if S is a countable intersection $S = \bigcap_{n\geq 1} U_n$, $U_n \subset X$ open.

(i) Show that any closed subset F of a metric space (M, d) is a G_δ *Hint:* consider the sets $F_r = \{x \in M; d(x, F) < r\}$.

(ii) Prove that if $S \subset X$ is a G_δ and is dense in X, then S is *generic* (that is, S is a countable intersection of open dense sets.)

7. If $f : X \to Y$ is any function (X topological space, Y metric space), the set of continuity $C_f \subset X$ is a G_δ set.

Hint: Let U_n be the union of all open sets V such that the diameter of $f(V)$ is less than $1/n$. Show $C_f = \bigcap_{n\geq 1} U_n$.

8. (i) $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$ (hence there are no functions $f : \mathbb{R} \to \mathbb{R}$ with continuity set $C_f = \mathbb{Q}$).

(ii) The irrationals $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ are a G_δ set.

(iii) In a separable Baire space without isolated points, no countable dense subset is a G_δ.

9. Let $f_n : \mathbb{R} \to \mathbb{R}$ continuous. Suppose $f_n \to f$ pointwise on $\mathbb{R}$. Then f is continuous at uncountably many points of $\mathbb{R}$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that the derivative $g = f' : \mathbb{R} \to \mathbb{R}$ is the pointwise limit of a sequence of continuous functions. (As a consequence, the set of continuity of g is a dense G_δ set.)