NOTES ON MANIFOLDS

LECTURE 1. Differentiable manifolds, differentiable maps

Def: topological m-manifold. Locally euclidean, Hausdorff, second-countable space. At each point we have a *local chart.* (U, φ), where $\varphi : U \to R^m$ is a topological embedding (homeomorphism onto its image) and $\varphi(U)$ is an open ball in R^m.

Def: differentiable structure. An *atlas of class* C^r ($r \geq 1$ or $r = \infty$) for a topological m-manifold M is a collection $\mathcal{U}$ of local charts (U, φ) satisfying:
(i) The domains U of the charts in $\mathcal{U}$ define an open cover of M;
(ii) If two domains of charts $(U, \varphi), (V, \psi)$ in $\mathcal{U}$ overlap $(U \cap V \neq \emptyset)$, then the transition map:

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^r diffeomorphism of open sets in R^m.
(iii) The atlas $\mathcal{U}$ is maximal for property (ii).

Examples:
(i) $S^n \subset R^{n+1}$, the unit sphere: a compact manifold covered by two charts to R^n (stereographic projection from north, south poles.)

(ii) RP^n, real projective$n$-space: the space of one-dimensional subspaces of R^{n+1}. This has an atlas with $(n+1)$ charts. Equivalently, it is the quotient of S^n by the action of $\mathbb{Z}_2$ generated by the antipodal map (and hence is compact).

(iii) Products of manifolds.

(iv) Open subsets of manifolds. For example, the group of invertible matrices GL_n, an open subset of $\mathbb{R}^{n \times n}$.

Proposition. Let $\sim$ be an equivalence relation on a topological space S. Consider the quotient space $p : S \to S/\sim$, with the quotient topology. Then:
(i) If $S/\sim$ is Hausdorff, equivalence classes of points are closed subsets of S.
(ii) If $\sim$ is an open equivalence relation (*def:* the saturation of open subsets of S is open in S; equivalently, p is an open map), then $S/\sim$ is Hausdorff iff the *graph* of $\sim$ is a closed subset of $S \times S$.

Exercise 1.1: (i) Prove that if $\sim$ is open and S is Hausdorff, then equivalence classes of points are closed subsets of S.

(i) Prove that if $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is a basis for the topology of S, then $\{p(U_\alpha)\}_{\alpha \in A}$ is a basis for the topology of $S/\sim$.

Def: differentiable map between differentiable manifolds. $f : M^m \to N^n$ (continuous) is *differentiable at* p if for any local charts $(U, \varphi), (V, \psi)$ at p, resp. $f(p)$ with $f(U) \subset V$ the map $\psi \circ f \circ \varphi^{-1} = F : \varphi(U) \to \psi(V)$ is differentiable at $x_0 = h(p)$ (as a map from an open subset of R^m to R^n.)

$f : M \to N$ (continuous) is *of class* C^r if for any charts $(U, \varphi), (V, \psi)$ on M resp. N with $f(U) \subset V$, the composition $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is of class C^r (between open subsets of R^m, resp. R^n.)

A C^r map $f : M \to N$ is a *diffeomorphism of class* C^r if it is a homeomorphism onto N and the inverse map $f^{-1} : N \to M$ is also of class C^r.

LECTURE 2. Tangent space at a point, differential of a map, tangent bundle.

Let M^n be a differentiable manifold, let $p \in M$. To define the tangent space at p, consider the set $\mathcal{A}_p$ of charts in the atlas of M which are defined in a neighborhood of p. On $\mathcal{A}_p \times R^n$, introduce the equivalence relation;

$$(\varphi, v) \sim (\psi, w) \leftrightarrow d(\psi \circ \varphi^{-1})(\varphi(p))[v] = w, \text{ where } (U, \varphi), (V, \psi) \in \mathcal{A}_p, v, w \in R^n.$$

The equivalence class $X_p = [\varphi, v]$ is a *tangent vector* at p; the space of tangent vectors at p is T_pM, the tangent space at p.

Observing that $\alpha[\varphi, v] + \beta[\varphi, w] = [\varphi, \alpha v + \beta w]$ is well-defined, we see that T_pM is a real vector space, with origin $0_p = [\varphi, 0]$ for any $(U, \varphi) \in \mathcal{A}_p$. For any $(U, \varphi) \in \mathcal{A}_p$, the map $v \mapsto [\varphi, v]$ defines a linear isomorphism $R^n \to T_pM$.

A tangent vector $X_p \in T_pM$ induces a linear operator from the space of smooth functions on M to $\mathbb{R}$:

$$\hat{X}_p : C^\infty_M \to \mathbb{R}, \quad \hat{X}_p(f) = d(f \circ \varphi^{-1})(\varphi(p))[v] \text{ if } X_p = [\varphi, v].$$

(That is, $\hat{X}_p$ is the directional derivative of f in the direction X_p.)

Exercise 2.1 Prove the following:

(i) $\hat{X}_p$ is well-defined, that is, it depends only on the equivalence class $X_p = [\varphi, v]$, not on the choices of φ or v;

(ii) $\hat{X}_p(f)$ depends only on f in a neighborhood of p. That is, if we take $g \in C^\infty_M$ and there exists $U \subset M$ (open neighborhood of p) so that $f_{|U} = g_{|U}$, then $\hat{X}_p(g) = \hat{X}_p(f)$.

(iii) (Leibniz rule) $\hat{X}_p(fg) = f(p)\hat{X}_p(g) + g(p)\hat{X}_p(f)$.

(iv) (differential of a map–see below) $df(p)[X_p] = Y_q, q = f(p) \Rightarrow \hat{Y}_q(g) = \hat{X}_p(g \circ f)$.

Thus $X_p \mapsto \hat{X}_p$ embeds T_pM as an n-dimensional subspace of $\mathcal{L}(C^\infty_M, \mathbb{R})$.

Let $F : M^m \to N^n$ be a differentiable map; let $p \in M, q = f(p) \in N$. The *differential of* f at p is the linear map $T = df(p) \in \mathcal{L}(T_pM, T_qN)$ defined via:

$$X_p = [\varphi, c] \Rightarrow T[X_p] = Y_q = [\psi, d(\psi \circ f \circ \varphi^{-1})(\varphi(p))[v]], \text{ if } (U, \varphi) \in \mathcal{A}^M_p, (V, \psi) \in \mathcal{A}^N_q, f(U) \subset V.$$

Exercise 2.2. Show that this is well-defined, that is, considering instead $(U_1, \varphi_1) \in \mathcal{A}^M_p, v_1 \in R^n$ so that $[\varphi, v] = [\varphi_1, v_1]$ and $(V_1, \psi_1) \in \mathcal{A}^N_q$ with $f(U_1) \subset V_1$, we get the same Y_p.

The *tangent bundle* is defined as the union of all tangent spaces, that is (omitting the 'hat' for the directional derivative from now on):

$$TM = \{(p, X) \in M \times \mathcal{L}(C^\infty_M, \mathbb{R}); X \in T_pM\}.$$

(Recall we regard T_pM as a subspace of $\mathcal{L}(C^\infty_M, \mathbb{R})$.) Define the projection: $\pi : TM \to M, \pi(p, X) = p$. To endow TM with a topology, we (a) require π to be continuous, and (b) consider the map, associated with a local chart (U, φ):

$$\hat{\varphi} : \pi^{-1}(U) \to U \times R^n, \hat{\varphi}(p, X_p) = (\varphi(p), v) \text{ if } X_p = [\varphi, v].$$

It is easy to see the maps $\hat{\varphi}$ are bijective. Hence there exists a unique topology on TM making each $\hat{\varphi}$ (for every $(U, \varphi) \in \mathcal{A}$) a homeomorphism. This topology endows TM with the structure of a (Hausdorff, second-countable) topological manifold of dimension $2n$.

Differentiable structure. Let $(U, \varphi), (V, \psi)$ be overlapping charts for M, $U \cap V \neq \emptyset$. So $F = \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a C^r diffeomorphism. Then the associated charts for TM also overlap, with the transition diffeomorphism (of class C^{r-1}):

$$\tilde{F} = \hat{\psi} \circ \hat{\varphi}^{-1} : \varphi(U \cap V) \times R^m \to \psi(U \cap V) \times R^m.$$

$$(x, v) \mapsto (F(x), dF(x)[v]).$$

Thus the C^r atlas on M induces a C^{r-1} atlas on TM. The projection $\pi : TM \to M$ is then a C^{r-1} map of manifolds.

Exercise 2.3. Show that $ker(d\pi(p, X)) = T_pM$. (Where we regard T_pM as a subspace of $T_{(p,X)}TM$. Why can we do this?)

LECTURE 3. Differentiable vector bundles; Grassmannians.

The tangent bundle is a special case of an important class of manifolds.

Def. A *locally trivial vector bundle* with typical fiber $\mathbb{V}$ is a triple (E, M, π), where E and M are differentiable manifolds and $\pi : E \to M$ a differentiable map, satisfying:

(i) Any $p \in M$ admits a neighborhood $U \subset M$ and a diffeomorphism ('local trivialization') $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{V}$ commuting with the identity on U: $\pi(e) = p_1(\Phi_U(e))$, where $p_1 : U \times \mathbb{V} \to U$ is the first projection.

(ii) If two trivializing domains overlap ($U \cap W \neq \emptyset$), the transition map has the form:

$$\Phi_W \circ \Phi_U^{-1} : U \times \mathbb{V} \to W \times \mathbb{V}, \quad (x, v) \to (x, T(x)v),$$

where $T : U \cap W \to GL(\mathbb{V})$ is a differentiable map.

E is *trivial* if there is a global diffeomorphism $\Phi : E \to M \times \mathbb{V}$, $\pi(e) = p_1(\Phi(e))$, so that $\Phi_{|E_p} \in \mathcal{L}(E_p, \mathbb{V})$, $E_p = \pi^{-1}(p)$.

Def. A *section* of E is a differentiable map $X : M \to E$ such that $\pi \circ \sigma = id_M$, i.e. $X(p) \in E_p$. A *vector field* on M is a section of TM: $X(p) \in T_pM$.

Exercise 3.1: E is trivial $\Leftrightarrow \exists$ a differentiable basis of sections $X_1, \ldots, X_n$ ($n = \dim \mathbb{V}$), i.e. $\{X_i(p)\}$ is a basis of E_p, for all $p \in M$.

Example 1: TS^1 is trivial: $V(x, y) = (-y, x)$ is a nonvanishing tangent vector field onS^1.

Example 2: TS^2 is not trivial: there is no non-vanishing tangent vector field on S^2. This follows from the following facts:

(i) If $f, g, \in C(S^n, S^n)$ and $f(x) + g(x) \neq 0 \forall x \in S^n \subset R^{n+1}$, then f is homotopic to g.

(ii) In particular, if $f \in C(S^2, S^2)$ has no fixed points, f is homotopic to α, the antipodal map α.

(iii) The identity map of S^2 is not homotopic to the antipodal map (this is true for even-dimensional spheres). The proof uses the homotopy invariance of the degree of a map.

(iv) Suppose V were a nonvanishing vector field on S^2. Consider the map: $f(x) = \frac{x+V(x)}{||x+V(x)||}$. Then f is homotopic to the identity (replace V by $tV, t \in [0,1]$). On the other hand, f has no fixed points, hence is homotopic to α. Contradiction!

Exercise 3.2. Prove (i), (ii) and (iv) in detail. In fact, prove the more general version of (i): if f, g satisfy $|f(x) + g(x)| < 2\ \forall x \in S^n$, then f is homotopic to g.

Exercise 3.3. (i) Generalizing Example 1, show that any odd-dimensional sphere admits a non-vanishing vector field. (*Hint.* Consider S^{2n+1} as the unit sphere in $\mathbb{C}^n$, complex n-dimensional space:

$$S^{2n+1} = \{(z_1, \ldots, z_n); \sum_j |z_i|^2 = 1, z_j = u_i + iv_j.\}.$$

(ii) Show that the 3-sphere is *parallelizable* (i.e., the tangent bundle TS^3 is trivial). *Hint:* Recall S^3 can be thought of as the set of unit quaternions (denote the quaternion algebra by $\mathbb{H}$). Use quaternion algebra and trial and error to find three l.i. tangent vector fields at a point $x + yi + zj + wk \in S^3 \subset \mathbb{H}$.

In addition to the tangent bundle of a manifold and trivial bundles, here are other common examples.

Exercise 3.4. (Normal bundle) Let $M^n \subset \mathbb{R}^N$ be a submanifold of euclidean space (*submanifold* is defined in the next lecture.) Consider the set:

$$NM = \{(p, v) \in M \times \mathbb{R}^N; v \in (T_pM)^\perp\}.$$

(Here we regard T_pM as a subspace of $\mathbb{R}^n$, and $\perp$ denotes orthogonal complement in $\mathbb{R}^N$. That is, we let $T_pM = d\psi(0)[R^n]$, where $\psi : U_0 \to \mathbb{R}^N$, $U_0 \subset \mathbb{R}^n$ open, $\psi(0) = p$, is a local parametrization of M, the inverse of a submanifold chart at p). Show that NM is a smooth vector bundle of rank (dimension of the fiber) equal to $N - n$.

Exercise 3.5. (Pullback bundle.) Let $\pi : E \to N$ be a vector bundle of rank n, $f : M \to N$ a smooth map. Consider the set:

$$F = \{(x, e) \in M \times E; f(x) = \pi(e)\}.$$

Show that, with the projection map $p(x, e) = x, p : F \to M$ has the structure of a (smooth) vector bundle with base M and n-dimensional fibers.

LECTURE 4. Submanifolds, immersions and embeddings.

Recall the *Inverse Function Theorem* in euclidean space:

Theorem. Let $U \subset R^m$ open, $f : U \to R^m$ be a C^r map ($r \geq 1$). Suppose $df(x) \in \mathcal{L}(R^m)$ is an isomorphism, for some $x \in U$. Then there exist open neighborhoods $U_1 \subset U$ of x, V of $f(x)$ in R^m, such that:
(i) f is a homeomorphism from U_1 onto V;
(ii) $f^{-1} : V \to U_1$ is of class C^r, with $df^{-1}(f(x)) = (df(x))^{-1} \in \mathcal{L}(R^m)$.

Thus if $df(x)$ is an isomorphism for all $x \in U$, then f is a C^r local diffeomorphism (a C^r local homeomorphism with C^r local inverses.)

Since this theorem is local, the extension to differentiable maps between manifolds is immediate; the statement will be left to the reader.

Here we recall:

Def. A C^r map $f : M \to N$ of C^r manifolds is an *immersion* if $df(p)$ has rank $m = \dim M$, for all $p \in M$ (in particular, $m \leq n$; usually we'll assume $m < n$ when referring to immersions.)

Up to taking local charts on the domain and range, immersions have a very simple form:

Theorem (local form of immersions.) Let $f : M^m \to N^n$, $m < n$, be a C^k map ($k \geq 1$); let $p \in M$ be such that $df(p)$ has rank m. Then there exist a local chart (V, φ) at p, complementary subspaces $E, F \subset R^n$ of dimensions $m, n-m$ (resp.), a neighborhood Z of $f(p)$ in N with $f(V) \subset Z$ and a C^k local chart for N at $f(p)$, $\psi : Z \to R^n = E \oplus F$ so that, with respect to this splitting:

$$\Phi = \psi \circ f \circ \varphi^{-1} : \varphi(V) \to R^n = E \oplus F \text{ has the form } \Phi(x) = (x, 0) \quad \forall x \in \varphi(V).$$

Proof. (Outline.) We consider the euclidean case, since the manifolds statement is easily reduced to it. So $f : U \to R^n$, with $U \subset R^m$ open. (In this case we don't need φ, and $V \subset U$ will be a neighborhood of $p \in U$.)The idea is to reduce the result to the inverse function theorem. So let $E = df(p)[R^m] \subset R^n$ (an m-dimensional subspace) and let $F = E^\perp$, the orthogonal complement to E (any subspace of R^n complementary to E would work), so $R^n = E \oplus F$. Now define:

$$h : U \times F \to R^n, \quad h(x, y) = f(x) + y, \quad h(p, 0) = f(p),$$

$$dh(p, 0) \in \mathcal{L}(R^m \oplus F; E \oplus F) \text{ invertible: } dh(p, 0)[u \oplus v] = df(p)[u] \oplus v.$$

By the IFT, h is a diffeomorphism from a neighborhood $V \times W \subset U \times F$ of $(p, 0)$ $(p \in V, 0 \in W)$ to a neighborhood $Z = h(V \times W) \subset R^n$ of $f(p)$. (In particular, $f(V) = h(V \times \{0\}) \subset Z$.) Let $\psi = h^{-1} : Z \to V \times W$. Then since $h(x, 0) = f(x)$, we have, for all $x \in V$:

$$\Phi(x) = (\psi \circ f)(x) = h^{-1}(f(x)) = (x, 0),$$

as claimed.

At this point we introduce two important definitions.

Definition. A subset $Z \subset N$ of an n-manifold N is a *submanifold* (of dimension $m \leq n$) if either (i) $m = n$ and Z is open in N; or (ii) $m < n$ and for each $p \in Z$ there exists a chart $\varphi : U \to R^n = R^m \times R^{n-m}$ at p in the atlas of N, such that $\varphi(U \cap Z) \subset R^m \times \{0_{n-m}\}$.

It is easy to see that where two of these 'submanifold charts' $(U, \varphi), (V, \psi)$ overlap, the coordinate change $\psi \circ \varphi^{-1} : R^m \times R^{n-m} \to R^m \times R^{n-m}$ preserves the splitting, and its restriction to a map $R^m \times 0_{n-m} \to R^m \times 0_{n-m}$ defines a coordinate change for a differentiable atlas of Z, namely:

$$\mathcal{A}^Z = \{(U \cap Z, \varphi); (U, \varphi) \in \mathcal{A}^M, \varphi(U \cap Z) \subset R^m \times 0_{n-m}\}.$$

Consider the *question:* suppose $f : M^m \to N^n$ is a differentiable immersion $(m < n)$, as well as injective. Is $f(M)$ a submanifold of N? In general this is false. (Example: an injective immersion from R to R^2 whose image includes the 'topologist's sine curve', accumulating on another arc of the image. An even more extreme example is an irrational line on the torus $T^2 = S^1 \times S^1$, as the image of an injective immersion $R \to T^2$)

Exercise 4.1. Use the local form of immersions to show that if $f : M^m \to N^n$ is an injective immersion $(m < n)$, then for any $p \in M$ there exists a neighborhood $V \subset M$ of p such that the image $f(V)$ is a submanifold of N. (But *careful!* $f(V)$ need not be open in $f(M)$, as the examples above show.)

Definition: A differentiable map $f : M^m \to N^n$ is an *embedding* into N if it is an injective immersion, and a homeomorphism onto its image $f(M)$ (with the induced topology as a subspace of N). Equivalently, if f is an injective immersion *and* an open map from M to $f(M)$ (with the topology induced from N.) Note that since f is then bijective from M to $f(M)$, it is equivalent to require f to be a closed map from M to $f(M)$.

Exercise 4.2. Show that if $f : M \to N$ is an embedding, $f(M)$ is an m-dimensional submanifold of N (use Exercise 4.1).

Example. If M is compact, then any injective immersion $f : M^m \to N^n$ $(m < n)$ is an embedding, and $f(M)$ is a submanifold of N. The reason is that a continuous, bijective map from a compact Hausdorff space X onto a (necessarily compact) Hausdorff space Y is automatically a homeomorphism (recall the simple proof.)

When M is non-compact, a useful sufficient condition is given by the following.

Definition. A continuous map $f : X \to Y$ (Hausdorff topological spaces, compactly generated) is a *proper map* if the preimage under f of any compact subset of Y is a compact subset of X. If X, Y are second-countable, this is equivalent to: if (x_n) is a divergent sequence in X, then $f(x_n)$ is a divergent sequence in Y. (The proof of equivalence is easy, and you should think about it; recall (x_n) divergent in X means given any $K \subset X$ compact, there exists an $N \geq 1$ such that $x_n \notin K$, for all $n \geq N$.)

Remark: Another definition found in the literature requires preimages of *points* to be compact. This is equivalent to the more common definition above, if X and Y are compactly generated.

Exercise 4.3. If $f : X \to Y$ is a proper map, and a local homeomorphism *onto* Y, then f is a covering map. Thus the notion 'proper map' bridges the gap between 'surjective local homeomorphism' and 'covering map'.

Exercise 4.4. Prove that a proper map $f : X \to Y$ is a closed map (takes closed subsets of X to closed subsets of Y).

It follows from this exercise (and 4.1, 4.2) that if $f : M^m \to N^n$ is a differentiable injective immersion *and* a proper map, then f is an embedding, and $f(M)$ is a submanifold of N.

However, it is easy to give examples of embeddings with image a submanifold, and which are not proper maps (for instance, the inclusion map of $(-1, 1)$ into $\mathbb{R}$, or an embedded curve spiraling to the unit circle from the inside.) These examples also show submanifolds of R^N need not be closed subsets of R^N. Also, a closed subset of R^n may be the image of a smooth injective immersion from some R^m $(m < n)$, and yet not be a submanifold of R^n. (Example?)

LECTURE 5. Submersions and regular values.

Theorem 5.1–local form of submersions. Let $f : M^m \to N^n$ $(m > n)$ be a C^k map between differentiable manifolds, $k \geq 1$. Let $p \in M$ such that $df(p)$ is a surjective linear map. Then for any splitting $T_pM = E \oplus F$, with $E = ker(df(p))$, $\dim(E) = m-n, \dim(F) = n$, there exist local charts (U, φ) for M at p, (V, ψ) for N at $q = f(p)$, $f(U) \subset V$, so that, with respect to the splitting:

$$\varphi : U \to W \times V_1, \psi : V \to F, \varphi(p) = 0 \in E, \psi(q) = 0 \in F, W \subset E, V_1 = \psi(V) \subset F,$$

we have:

$$\psi \circ f \circ \varphi^{-1}(x, y) = y, \quad x \in W, y \in V_1.$$

Proof. Since the result is local, it suffices to consider the case where M is an open subset of R^m and N is an n-dimensional real vector space. (Then the chart ψ won't be needed, and we may take $F = N$ and $V_1 = V$). Choose coordinates in R^m so that $p = 0$. So $df(0) \in \mathcal{L}(R^m; N)$ is onto. With $E = ker(df(0))$, we fix the splitting $R^m = E \oplus N$ and define:

$$\varphi : M \to E \oplus N, \quad \varphi(x) = pr_E x \oplus f(x), \quad \varphi(0) = f(0) = q \in N.$$

The differential of φ at 0 is:

$$d\varphi(0)[v] = pr_E[v] \oplus df(0)[v], \quad v = pr_E[v] \oplus pr_N[v] \in R^m,$$

and it is easy to see this has trivial kernel, since $df(0)$ is an isomorphism on N. By the inverse function theorem, there exist neighborhoods $U \subset M$ of 0, $V \subset N$ of q, $W \subset E$ of 0 so that $\varphi : U \to W \times V$ is a diffeomorphism. Considering now, for $(x, y) \in W \times V$:

$$(x, y) = \varphi(\varphi^{-1}(x, y)) = (pr_E(\varphi^{-1}(x, y)), f(\varphi^{-1}(x, y)),$$

we see that $(f \circ \varphi^{-1})(x, y) = y$, as we need to show.

Definitions. Let $f : M^m \to N^n$ be a differentiable map between manifolds, with $m \geq n$. A point $c \in N$ is a *regular value* for f if either (i) $f^{-1}(c) = \emptyset$; or (ii) for any $p \in f^{-1}(c)$, $df(p) \in \mathcal{L}(T_pM, T_{f(q)}N)$ is surjective. ($c \in N$ is called 'critical value' if it is not a regular value; $p \in M$ is a 'critical point' if $rank(df(p)) < m$).

f is a *submersion* if $df(p)$ is surjective, for all $p \in M$ (thus every $c \in N$ is a regular value.) From the above, one easily shows:

Proposition 5.2. Let $f : M^m \to N^n$ be C^k map of C^k manifolds, $m \geq n$. Let $c \in N$ be a regular value of f , set $L_c = f^{-1}(c)$. Then L_c is a C^k submanifold of M, of dimension $m-n$, with tangent space $T_pL_c = ker(df(p))$ if $p \in L_c$.

Exercise 5.1. Prove: any submersion is an open map.

Exercise 5.2. Prove the first statement in the proposition, by arguing that the local form of immersions gives submanifold charts at any $p \in L_c$.

It is not hard to prove the last statement of Prop. 5.2 (identifying the tangent space at $p \in L_c$.) Let $X_p \in T_pL_c$; consider the submanifold chart at p:

$$\varphi = (\varphi_1, \varphi_2) : U \to W \times V, \quad \varphi_2 = 0 \text{ on } U \cap L_c$$

given by Theorem 5.1. So $(U \cap L_c, \varphi_1)$ is a local chart for L_c at p. In the notation of Lecture 2, $X_p = [\varphi_1, v]$ for some $v \in R^{m-n}$. Now:

$$df(p)[X_p] = [\psi, d(\psi \circ f \circ \varphi_1^{-1})(0)[v]] = [\psi, 0] = 0_{T_pM},$$

since $\psi \circ f \circ \varphi_1^{-1}(x) = 0$, for all $x \in W$. Thus $T_pL_c \subset ker(df(p))$, and since they have the same dimension, they must coincide.

Example 1. Let $M_n = R^{n^2}$ be the space of $n \times n$ real matrices, $det : M \to R$ the determinant function. Then for any $0 \neq c \in R$, the level set $\{X \in M_n; det(X) = c\}$ is a submanifold of M_n, of codimension 1. In particular, the matrix group SL_n (corresponding to $c = 1$) is a codimension 1 submanifold of R^{n^2}. (It is enough to show any $c \neq 0$ is a regular value of det.)

Example 2. The orthogonal group $O(n) = \{X \in M_n; XX^T = I_n\}$ is a *compact* submanifold of codimension $n(n+1)/2$ in M_n. This follows from the fact the identity matrix I_n is a regular value of the map $X \mapsto XX^T$ from M_n to Sym_n, the linear space of symmetric $n \times n$ matrices.

Example 3. The set of $m \times n$ matrices of rank r is a submanifold of R^{mn} of codimension $(m-r)(n-r)$. ([G-P, p. 27])