Lemma

$$\delta\varphi = -\sum_{i=1}^{n} \iota(e_i)(\nabla_{e_i}\varphi), \quad \varphi \in \Omega^p$$

where (e_i) is an orthonormal frame.

Proof. Let (θ^i) be the dual co-frame of 1-forms. At a fixed $x \in M$, we assume (e_i) is chosen so that $\nabla_{e_j} e_i|_x = 0$ (and hence $\nabla_{e_j}\theta^i|_x = 0$).

Since $\delta\varphi = (-1)^{np+n+1} * d * \varphi$, we compute $d*\varphi$ for $\varphi = a\,\theta_1 \wedge \ldots \wedge \theta_p$ (Here $a \in C^\infty$ is a function; it's enough to consider this φ.)

$$*\varphi = a\,\theta_{p+1} \wedge \ldots \wedge \theta_n, \quad \text{(recall } \varphi \wedge *\varphi = |\varphi|^2 \text{vol}_g,\ \text{vol}_g = \theta_1 \wedge \ldots \wedge \theta_n).$$

We have, at x:

$$d * \varphi = \sum_{j=1}^{p} e_j(a)\,\theta_j \wedge (\theta_{p+1} \wedge \ldots \wedge \theta_n).$$

$$*(d * \varphi) = \sum_{j=1}^{p} e_j(a) * (\theta_j \wedge \theta_{p+1} \wedge \ldots \wedge \theta_n)$$

where $*(\theta_j \wedge \theta_{p+1} \wedge \ldots \wedge \theta_n) = \sigma_j\,\theta_1 \wedge \ldots \wedge \hat{\theta}_j \wedge \ldots \wedge \theta_p$ ($\hat{\theta}_j$ means omit θ_j). Here $\sigma_j = \pm 1$, and is determined by:

$$(\theta_j \wedge \theta_{p+1} \wedge \ldots \wedge \theta_n) \wedge *(\theta_j \wedge \theta_{p+1} \wedge \ldots \wedge \theta_n) = \theta_1 \wedge \ldots \wedge \theta_n.$$

Hence:

$$(\theta_j \wedge \theta_{p+1} \wedge \ldots \wedge \theta_n) \wedge \sigma_j\,(\theta_1 \wedge \ldots \wedge \hat{\theta}_j \wedge \ldots \wedge \theta_p) = \theta_1 \wedge \ldots \wedge \theta_n.$$

First move all θ_q with $q > p$ to the right, starting with θ_n; since there are $n-p$ of them:

$$LHS = \sigma_j\,(-1)^{(n-p)(p-1)}\theta_j \wedge (\theta_1 \ldots \wedge \hat{\theta}_j \wedge \ldots \wedge \theta_p) \wedge (\theta_{p+1} \ldots \wedge \theta_n)$$

$$= \sigma_j(-1)^{(n-p)(p-1)+j-1}\,\theta_1 \wedge \ldots \wedge \theta_n.$$

We conclude $\sigma_j = (-1)^{(n-p)(p-1)+j-1}$. Thus,

$$\delta\varphi = (-1)^{np+n+1} \sum_{j=1}^{n} e_j(a)\,\sigma_j\,\theta_1 \wedge \ldots \wedge \hat{\theta}_j \wedge \ldots \wedge \theta_p$$

$$= (-1)^{np+n+1} \sum_{j=1}^{n} e_j(a)\,\sigma_j(-1)^{j-1}\iota(e_j)(\theta_1 \wedge \ldots \wedge \theta_p)$$

Note
$$(-1)^{np+n+1}\sigma_j(-1)^{j-1} = (-1)^{(n-p)(p-1)}(-1)^{np+n+1}$$
and
$$(n-p)(p-1)+np+n+1 = np-n-p^2+p+np+n+1 \equiv -p^2+p+1 \equiv 1 \mod (2).$$
Hence:
$$\delta\varphi = -\sum_{j=1}^{p} e_j(a)\,\iota(e_j)(\theta_1 \wedge \ldots \wedge \theta_p)$$
$$= -\sum_{j=1}^{n} \iota(e_j)\nabla_{e_j}(a\theta_1 \wedge \ldots \wedge \theta_p) = -\sum_{j=1}^{n} \iota(e_j)(\nabla_{e_j}\varphi)$$
(at x), as claimed. □