Yau's theorem on positive harmonic functions in nonnegative Ricci curvature.

Theorem. Let M be a complete, noncompact Riemannian manifold with the lower Ricci curvature bound $Ric(X,X) \geq -(n-1)a^2|X|^2$. Let f be a positive harmonic function on M. Then we have the bound, pointwise on M:

$$\frac{|df|}{f} \leq \sqrt{2}(n-1)a.$$

In particular, if $Ric \geq 0$ on M, then f is constant.

Proof. [1] Let $e_f = |df|^2$, the 'energy density' of f. Fix an arbitrary $x_0 \in M$ and any $R > 0$. Denote by r the distance function to x_0. In a closed ball $B_R = B_R(x_0)$ where f is non-constant, consider the continuous function:

$$\Phi(x) = (R^2 - r(x)^2)^2 \frac{e_f}{f^2}(x).$$

Φ is nonnegative, continuous on B_R and zero on ∂B_R, and hence attains an interior, positive global max at some point $z \in int(B_R)$ (or else is identically zero, and hence $e_f \equiv 0$, a contradiction.)

If r were smooth in a neighborhood of z, we would now consider the conditions $d\log\Phi(z) = 0$, $\Delta\log\Phi(z) \leq 0$. But r may fail to be smooth near z, for instance if z is in the cut locus of x_0. We're interested in these conditions at z only, and a key idea is to replace r by a smooth function ρ, locally near z.

Lemma 1. There exists a neighborhood $W \subset int(B_R)$ of z (an open geodesic ball with center z) and a positive smooth function $\rho : W \to \mathbb{R}$ so that: (i) $\rho(x) \geq r(x)$ in W and $\rho(z) = r(z)$; (ii) $|d\rho|(z) = 1$; (iii) $\rho\Delta\rho(z) \leq (n-1)a\rho(z)\coth a\rho(z)$.

Using ρ, we consider the smooth function of $x \in W$:

$$\Phi_0(x) = (R^2 - \rho(x)^2)^2 \frac{e_f}{f^2}(x).$$

Note that, from condition (i) in the lemma, we may assume (taking a smaller W if needed) $\rho(x) \leq R$ for $x \in \overline{W}$; so $0 < R^2 - \rho(x)^2 \leq R^2 - r(x)^2$ for $x \in W$, and thus $\Phi_0(x) \leq \Phi(x) \leq \Phi(z) = \Phi_0(z)$ for $x \in \overline{W}$; so z is an interior maximum point for Φ_0 in W, and the argument may proceed.

The critical point condition $d\log\Phi_0(z) = 0$ yields:

$$-\frac{4\rho d\rho}{R^2-\rho^2} + \frac{de_f}{e_f} - 2\frac{df}{f} = 0, \text{ so } (\frac{de_f}{e_f} - 2\frac{df}{r})(z) = \frac{4\rho d\rho}{R^2-\rho^2}(z).$$

[1] Essentially the proof given in [Wu, pp. 71-75], except for minor changes of notation and rearrangement.

The Laplacian condition at z is:

$$\Delta \log(R^2-\rho^2)^2 + \{\frac{\Delta e_f}{e_f} - \frac{|de_f|^2}{e_f^2} + 2\frac{e_f}{f^2}\} \leq 0 \text{ at } z,$$

where the assumption $\Delta f = 0$ has been used in computing the last term. To estimate the second term (in curly brackets) we recall the Bochner formula for the harmonic 1-form df:

$$\Delta e_f = \Delta |df|^2 = 2|D^2 f|^2 + 2Ric(\nabla f, \nabla f) \geq 2|D^2 f|^2 - 2(n-1)a^2 e_f,$$

where $D^2 f = \nabla df$ is the Hessian of f, and appeal to the following 'calculus lemma':

Lemma 2. Let f be a non-constant harmonic function on a Riemannian manifold M. Then the norm squared of the Hessian of f may be estimated (at each non-critical point of f) in terms of the norm squared of the differential of its energy density $e_f = |df|^2$:

$$|D^2 f|^2 \geq (\frac{1}{4} + \frac{1}{8(n-1)})\frac{|de_f|^2}{e_f}.$$

(Note that at a critical point of f, the statement of the theorem is trivial.) Combining Lemma 2 and the Bochner formula, we find:

$$\frac{\Delta e_f}{e_f} - \frac{|de_f|^2}{e_f^2} \geq -(\frac{1}{2} - \frac{1}{4(n-1)})\frac{|de_f|^2}{e_f^2} - 2(n-1)a^2.$$

Next we need to bound (at the critical point z) $\frac{|de_f|^2}{e_f^2}$ from above, in terms of $\frac{|df|^2}{f^2} = \frac{e_f}{f^2}$ .To do this we use the critical point condition at z, combined with the elementary inequality $(a+b)^2 \leq (1+\frac{1}{t})a^2 + (1+t)b^2$ (for any $t > 0$):

$$\frac{|de_f|^2}{e_f^2} = |2\frac{df}{f} + \frac{4\rho d\rho}{R^2-\rho^2}|^2 \leq 4\frac{e_f}{f^2}(1+\frac{1}{t}) + \frac{16R^2}{(R^2-\rho^2)^2}(1+t) \text{ at } z,$$

where we also used the facts $\rho(z) = r(z) \leq R$ and $|d\rho|(z) = 1$ (lemma 1.)

Putting all of this together and rearranging we find, for the term in curly brackets in the Laplacian condition at z, the estimate:

$$\frac{\Delta e_f}{e_f} - \frac{|de_f|^2}{e_f^2} + 2\frac{e_f}{f^2} \geq A(t)\frac{e_f}{f^2} - (\frac{1}{2} - \frac{1}{4(n-1)})\frac{16R^2}{(R^2-\rho^2)^2}(1+t) - 2(n-1)a^2,$$

where:

$$A(t) = -\frac{2}{t} + \frac{1}{n-1}(1+\frac{1}{t}).$$

Note that one easily finds $T > 0$ depending only on n, so that $A(t) > 0$ if $t > T$; and also that $A(t) \to \frac{1}{n-1}$ as $t \to \infty$.

To estimate (from below, at z) the first term in the Laplacian condition at z, we appeal to the following:

Corollary of Lemma 1. At the local max z of Φ (or of Φ_0), we have:

$$\Delta \log(R^2 - \rho^2)^2 \geq -\frac{4nR^2(2 + Ra)}{(R^2 - \rho^2)^2}.$$

Thus, at z we have:

$$A(t)\frac{e_f}{f^2} \leq \frac{4R^2}{(R^2 - \rho^2)^2}[n(2 + Ra) + (2 - \frac{1}{n-1})(1 + t)] + 2(n-1)a^2.$$

Now use the easy upper bound:

$$4R^2[n(2 + Ra) + (2 - \frac{1}{n-1})(1 + t)] \leq 4R^2[n(2 + Ra) + 2(1 + t)] := \gamma(R, t).$$

So at z: $A(t)\frac{e_f}{f^2} \leq 2(n-1)a^2 + \gamma(R^2 - \rho^2)^{-2}$. We conclude that, at z (assuming $t > T$):

$$\Phi_0(z) = (R^2 - \rho^2)^2 \frac{e_f}{f^2}(z) \leq \frac{1}{A(t)}[2(n-1)a^2(R^2 - \rho^2)^2 + \gamma(R, t)].$$

Since z is a global maximum point for Φ in $B_R = B_R(x_0)$ and $\rho(z) = r(z)$, we have:

$$R^4 \frac{e_f}{f^2}(x_0) = \Phi(x_0) \leq \Phi(z) = \Phi_0(z) \leq \frac{1}{A(t)}[2(n-1)a^2(R^2 - r(z)^2)^2 + \gamma(R, t)].$$

Dividing both sides by R^4 and noting $(R^2 - r(z)^2)^2 R^{-4} \leq 1$, we have:

$$\frac{e_f}{f^2}(x_0) \leq \frac{1}{A(t)}[2(n-1)a^2 + \frac{\gamma(R, t)}{R^4}].$$

Now fix $t > T$ and take limits as $R \to \infty$. Since the dependence of $\gamma(R, t)$ on R is cubic, the term $\gamma(R, t)R^{-4}$ vanishes in the limit, and we have:

$$\frac{e_f}{f^2}(x_0) \leq \frac{2(n-1)a^2}{A(t)}, \quad \forall t > T.$$

Taking limits now as $t \to \infty$, we conclude:

$$\frac{e_f}{f^2}(x_0) \leq 2(n-1)^2 a^2,$$

as we wished to show. □

Proof of Lemma 2. Fix $x_0 \in M$, a non-critical point of f, and pick a normal orthonormal frame at x_0, $\nu = e_1, e_2, \ldots, e_n$, where $\nu = \nabla f / |\nabla f|$ is normal to the level set of f at x_0, and $e_i(f) = 0$ at x_0 for $i \geq 2$. (Recall 'normal frame at x_0' means $\nabla_X e_i(x_0) = 0$, for all $X \in T_{x_0}M$.) In particular we have at x_0:

$$e_f = \nu(f)^2, \quad D^2_{e_i e_j} f_{|x_0} = e_i(e_j f)_{|x_0},$$

and with ($\theta^i, i = 1, \dots n$ the co-frame):

$$de_f = 2\sum_{i,j=1}^{n}(e_j f)e_i(e_j f)\theta^i = 2\nu(f)\sum_{i=1}^{n} D^2_{e_i,\nu} f\theta^i,$$

so at x_0:

$$|de_f|^2 = 4e_f \sum_{i=1}^{n}(D^2_{e_i,\nu}f)^2.$$

We estimate the norm squared of the Hessian at x_0 from below by leaving out the sum of terms $(D^2_{e_i,e_j}f)^2$ with $i \neq j$ and both greater than 1:

$$|D^2 f|^2 \geq \sum_{j=1}^{n}(D^2_{\nu,e_j}f)^2 + \sum_{j=2}^{n}(D^2_{e_j,e_j}f)^2 + \sum_{j=2}^{n}(D_{e_j,\nu}f)^2 = (a) + (b) + (c),$$

where for the first term: $(a) = \frac{|de_f|^2}{4e_f}$ at x_0. We estimate (b) using Cauchy-Schwarz and the assumption $\Delta f = 0$:

$$\sum_{j=2}^{n}(D^2_{e_j,e_j}f)^2 \geq \frac{1}{n-1}(\Delta f - D^2_{\nu,\nu}f)^2 \geq \frac{1}{n-1}(\frac{1}{2}(D^2_{\nu,\nu}f)^2 - (\Delta f)^2) = \frac{1}{2(n-1)}(D^2_{\nu,\nu}f)^2.$$

We combine these two estimates for (a) and (b) and multiply (c) by a factor smaller than one to conclude:

$$(a) + (b) + (c) \geq \frac{|de_f|^2}{4e_f} + \frac{1}{2(n-1)}[(D^2_{\nu,\nu}f)^2 + \sum_{j=2}^{n}(D^2_{e_j,\nu}f)^2],$$

and then notice the term in square brackets is again equal to (a), and therefore at x_0:

$$|D^2 f|^2 \geq (\frac{1}{4} + \frac{1}{8(n-1)})\frac{|de_f|^2}{e_f},$$

as we wished to show. $\square$

Proof of the Corollary to Lemma 1. From Calculus we have, at the critical point z of Φ_0:

$$\Delta \log(R^2 - \rho^2)^2 = -\frac{4\rho\Delta\rho}{R^2 - \rho^2} - \frac{4|d\rho|^2}{R^2 - \rho^2} - \frac{8\rho^2|d\rho|^2}{(R^2 - \rho^2)^2}$$

$$= -\frac{4}{R^2 - \rho^2}(\rho\Delta\rho + \frac{R^2 + \rho^2}{R^2 - \rho^2})$$

(Note $\rho(z) = r(z) < R$.) From lemma 1, we have $|d\rho|(z) = 1$ (already used above), and the estimate:

$$\rho\Delta\rho \leq (n-1)a\rho\coth(a\rho) \leq (n-1)(1+a\rho),$$

where we also used the Calculus estimate $1 \leq t\coth t \leq (1+t)$, $\forall t > 0$ (exercise.) Thus:

$$\rho\Delta\rho + \frac{R^2+\rho^2}{R^2-\rho^2} \leq \frac{1}{R^2-\rho^2}[(n-1)(R^2-\rho^2)(1+a\rho) + R^2 + \rho^2],$$

and we estimate the expression in square brackets as follows:

$$[\ldots] \leq nR^2(1+aR)+2R^2 = R^2[n(1+aR)+2] \leq R^2[n(1+aR)+n] = nR^2(2+aR),$$

and we conclude, as desired:

$$\Delta\log(R^2-\rho^2)^2 \geq -\frac{4nR^2(2+aR)}{R^2-\rho^2}.$$

□