**PRESENTATION TOPICS**–MATH 661, Fall 2024

**Paper:** *From vanishing theorems to estimating theorems: the Bochner technique revisited* by Pierre H. Bérard, Bulletin AMS, October 1988, p.371.

**TOPIC 1:** Present the proof of the following theorem (Thm V on p.379, with $\epsilon = -1$.)

*Theorem:* Let $M^n$, $n \geq 3$, be compact Riemannian without boundary, $E \to M$ a Riemannian vector bundle over $M$ of rank $l$, with metric connection $\nabla$. Suppose $\Delta_H$ is a second-order elliptic differential operator on smooth sections of $E$, satisfying pointwise the Weitzenböck formula:

$$\Delta_H s = \nabla^* \nabla s + \mathcal{R} s.$$

Denote by $\mathcal{H}$ the space of (smooth) sections of $E$ satisfying $\Delta_H s = 0$. Then for any $\alpha > 0$ there exists a constant $A = A(n, \alpha) > 0$ such that if $Ric_g \geq -(n-1)\alpha^2$ on $M$, we have:

$$\mathcal{R}_{min}(g)(diam_g(M))^2 \geq -A \Rightarrow \dim(\mathcal{H}) \leq l.$$

*Remark 1.* The model case is $E = \Lambda^p(T^*M)$, with section space $\Omega^p_M$, the $p$-forms on $M$, and $\Delta_H$ the Hodge Laplacian on $p$-forms. In this case the theorem generalizes the Gromoll-Meyer theorem, which hypothesizes positive curvature operator to conclude $\mathcal{H} = \{0\}$ (or nonnegative curvature operator to conclude $\dim(\mathcal{H}) \leq l$, or equivalently, harmonic $p$-forms are parallel.)

*Remark 2.* Taken as a whole, this survey paper is a mini-course on core topics of Geometric Analysis. But the organization and multiple paths make navigating it to extract a concise proof of a particular result time-consuming. So I'll introduce the background analysis material in the various appendixes in lecture, and the presenters should focus on the proof of this particular theorem, following the guide given below.

**ANALYTICAL PRELIMINARIES** (will be introduced in lecture.)

(i) Lower bound on Ricci curvature implies isoperimetric profile comparison (Appendix I.)
(ii) Sobolev embedding: constants based on the isoperimetric profile (Appendix VI, Theorem 3.)
(iii) Kato's inequalities (p. 380)
(iv) Moser iteration (Appendix V, Theorem 3, p. 395).

**Presentation:** The main theorem follows directly from Theorem 3 on p.389 (with $\lambda = 0$, $\epsilon = -1$, $\nu = \frac{n}{n-2}$) and Corollary 5. So the main goal is to present the proof of those, putting together the argument based on the analytical prerequisites introduced in advance. This is referred to as "first method" on p.380/381, where a brief outline of the proof is given. The 'first method' is based on 'P.Li's Lemma' (Lemma 1, Appendix II, p.387), the statement and proof of which (p. 387/388) should also be included in the presentation.

**Paper:** *Bandwidth and focal radius with positive isotropic curvature*, by Aaron Chow and Jingze Zhu (ArXiv, 30 May 2024.)

**PRELIMINARIES** (will be introduced in lecture):

(i) Hodge theory w/ Dirichlet and Neumann BCs (lemma 2.4, thm 2.5) for the twisted Hodge-de Rham complex
(ii) Integrated Weitzenböck formula with boundary terms, for the twisted Dirac operator (lemmas 3.2, 3.3/ Cor. 3.4)
(iii) Relating the boundary term in the Weitzenböck formula to the 2nd fundamental form of the boundary (lemma 3.6).

**TOPIC 2:** Vanishing of second betti number under positive isotropic curvature and 2-convexity of the boundary (Theorem A (i) in the paper.)

(i) Prop 3.5 (positive isotropic curvature and the curvature term on 2-forms): state without proof.
(ii) Lemma 3.7, Cor. 3.8: with proof. (2-convexity–more generally, a $-\delta$ lower bound– and control of the boundary terms.)
(iii) Prop 3.12 (i), with proof (implies Thm A(i) immediately, needing only $f = 0$ (no twisting) and $\delta = 0$).

**TOPIC 3:** Upper bound on the boundary width, under $b_2 \neq 0$ and lower bounds on Ricci and isotropic curvature and almost 2-convexity of the boundary (Thm B in the paper for $b_2$)

(i) Lemma 4.1(i): estimate of the Hessian term (of the twisting function $f$) in Prop. 3.12 (with proof.)
(ii) Proof of Thm. B (pp. 22/23) using the Laplacian comparison (B.5) in the Appendix: obtain (4.4) and explain how it leads to a contradiction.