

HOMOLOGY LONG EXACT SEQUENCE: GEOMETRIC APPROACH

Notation: The following applies equally to $A \subset X$ (nonempty subspace) in singular homology, or $A \subset X$ (subcomplex of a simplicial complex) in simplicial homology, so we use the notation $C_n(X), C_n(A), C_n(X, A) = C_n(X)/C_n(A)$ for chain groups. If z is a cycle or a relative cycle (in $Z_n(A), Z_n(X), Z_n(X, A)$) we denote by $[z]_A, [z]_X, [z]_{(X,A)}$ its homology class. $z \sim_A w, z \sim_X w, z \sim_{(X,A)} w$ denote ‘homologous to’ in each context, for instance for $z, w \in Z_n(X, A)$:

$$z \sim_{(X,A)} w \text{ means } z - w = \partial\alpha + \beta, \quad \alpha \in C_{n+1}(X), \beta \in C_n(A).$$

We also use $z \sim w \bmod A$ with the same meaning. Incidentally, we adopt the geometric definition of relative cycle (rather than look at quotient groups):

$$Z_n(X, A) = \{z \in C_n(X); \partial z \in Z_n(A)\}.$$

The connecting operator is defined as:

$$\partial_* : H_n(X, A) \rightarrow H_{n-1}(A), \quad [z]_{(X,A)} \mapsto [\partial z]_A, \quad z \in Z_n(X, A).$$

Finally, we say two chains in X are ‘congruent mod A ’ if they differ by a chain in A : $z \equiv w \bmod A$ means $z - w \in C_n(A)$.

Exactness at each step of the sequence has a ‘geometric’ content, which suggests a natural short proof.

$$\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \rightarrow \cdots$$

(i) *Exactness at $H_n(X)$:* $\ker(j_*) = \text{im}(i_*)$.

By definition, for a cycle $z \in Z_n(X)$, $j_*[z]_X = 0$ means $z \sim 0 \bmod A$ (that is, $z = \partial\alpha + \beta$, for some $\alpha \in C_{n+1}(X), \beta \in Z_n(A)$.)

On the other hand, $[z]_X = i_*[u]_A$ for some $u \in Z_n(A) \subset Z_n(X)$ iff $z \sim_X u$. Thus the geometric content of exactness at this step is:

(I) A cycle $z \in Z_n(X)$ is homologous (in X) to a cycle in A iff $z \sim 0 \bmod A$.

Proof. In one direction, this follows from the definition of $z \sim 0 \bmod A$ (as just seen).

In the other, let $\beta \in Z_n(A)$ be a cycle so that $i_*[\beta]_A = [z]_X$, or $z \sim_X \beta$. This means $z = \partial\alpha + \beta$ ($\alpha \in C_{n+1}(X), \beta \in Z_n(A)$), so $z \sim 0 \bmod A$.

(ii) *Exactness at $H_n(A)$:* $\ker(i_*) = \text{im}(\partial_*)$. Geometric content:

(II) A cycle in A bounds in X iff it is homologous in A to the boundary of a relative (X, A) cycle.

Proof: In fact, the following two sets are clearly equal:

$$\begin{aligned} \ker(i_*) &= \{[z]_A; z \in Z_n(A), z = \partial w \text{ for some } w \in C_{n+1}(X) \text{ with } \partial w \in Z_n(A)\}, \\ \text{im}(\partial_*) &= \{[\partial w]_A; w \in Z_{n+1}(X, A)\} = \{[\partial w]_A; w \in C_{n+1}(X) \text{ with } \partial w \in Z_n(A)\} \end{aligned}$$

(iii) *Exactness at $H_n(X, A)$* : $\ker(\partial_*) = \text{im}(j_*)$. The geometric content is:

(III) Let $z \in Z_n(X, A)$ be a relative cycle. Then $\partial z \sim_A 0$ iff there exists an absolute cycle $w \in Z_n(X)$ such that $z \equiv w \pmod A$.

Proof: If $\partial z = \partial u$ for some $u \in C_n(A)$, let $w = z - u \in Z_n(X)$. Then $z \equiv w \pmod A$.

Conversely, if there exist a cycle $w \in Z_n(X)$ and a chain $u \in C_n(A)$ such that $w = z - u$, then $\partial z = \partial u$, so $[\partial z]_A = 0$.

We may think of (I), (II), (III) as three independent statements relating relative homology to absolute homology, following directly from the definitions. In less precise wording:

(I) A cycle in X is homologous to a cycle in A iff it is homologous to zero mod A .

(II) A cycle in A bounds in X iff it is homologous in A to the boundary of a relative (X, A) cycle.

(III) A relative (X, A) cycle has boundary nullhomologous in A iff it is congruent mod A to an absolute cycle in X .

Exercise: *Long exact sequence of a triple.* Consider now an inclusion $A \subset V \subset X$ (topological spaces or subcomplexes of simplicial complexes, all assumed nonempty.) For instance, one could think of a submanifold A of a manifold X , with V a tubular neighborhood of A .

With definitions as before, we have the sequence of subgroups and inclusion homomorphisms:

$$Z_n(V, A) \xrightarrow{i_\#} Z_n(X, A) \xrightarrow{j_\#} Z_n(X, V),$$

and corresponding inclusion homomorphisms i_*, j_* in homology. The connecting homomorphism is defined as follows.

$$\partial_* : H_n(X, V) \rightarrow H_{n-1}(V, A), \quad [z]_{(X, V)} \mapsto [\partial z]_{(V, A)}, \quad z \in Z_n(X, V), \partial z \in Z_{n-1}(V),$$

where for $w, u \in Z_{n-1}(V)$:

$$w \sim_{(V, A)} u \Leftrightarrow w - u = \partial\alpha + \beta; \alpha \in C_n(V), \beta \in Z_n(A).$$

(Check that ∂_* is well-defined in homology.) The long exact sequence of the triple is:

$$\cdots \rightarrow H_n(V, A) \xrightarrow{i_*} H_n(X, A) \xrightarrow{j_*} H_n(X, V) \xrightarrow{\partial_*} H_{n-1}(V, A) \rightarrow \cdots$$

Exercise. Following the model given above for the long exact sequence of pairs, establish exactness at each of the three steps, including the ‘geometric content’ of exactness (analogously to (I),(II),(III) above).