PROBLEMS ON DE RHAM COHOMOLOGY[1]

1. Let N be a submanifold of the manifold M (with positive codimension.). Let $\alpha \in \Omega^r(M)$ and $\beta = \alpha_{|N}$.

(a) α is closed if, and only if, $d\beta(x) = 0$ for all $x \in N$. (Prove or give a counterexample.)

(b) α is exact if, and only if, for all submanifolds $N \subset M$ of codimension 1, the restriction $\beta = \alpha_{|N}$ is exact.

2. Denote by P_n the set of upper-triangular $n \times n$ matrices with positive diagonal entries. Prove that $GL(n)$ is diffeomorphic to the product of manifolds $P_n \times SO(n)$. Explain why this implies the de Rham cohomology spaces $H^r(GL(n))$ are finite-dimensional.

3. Let M be a contractible n-dimensional-manifold (that is, properly homotopy equivalent to a point.) Prove that $H^r_c(M) = 0$, for al $r \neq n$.

4. Let $T \subset \mathbb{R}^3$ be the torus of revolution, let M be the unbounded connected component of $\mathbb{R}^3 \setminus T$. Prove that $H^r(M) \approx \mathbb{R}$ for $r = 1, 2, 3$.

5. Let M be a compact, connected, oriented n-dimensional manifold. Prove that for any $m \in \mathbb{Z}$ there exists a continuous map $f : M \to S^n$ of degree m.

6. Let M be a compact, connected, oriented n-dimensional manifold. If there exists an r with $0 < r < n$ and $H^r(M) \neq 0$, prove that any smooth map $f : S^n \to M$ has degree 0.

7. *Relative de Rham cohomology.* Let $N \subset M$ be a compact submanifold of positive codimension. Denote by $\Omega^r(M; N)$ the vector space of r-forms ω on M with compact support, such that $\omega_{|N} = 0$ (that is, $\omega(x)(v_1, \dots, v_r) = 0$ if $x \in N, v_1, \dots, v_r \in T_xN$.).

Denote by $H^r_c(M; N)$ the r-dimensional cohomology space of the complex:

$$\dots \to \Omega^r_c(M; N) \to \Omega^{r+1}_c(M; N) \to \dots$$

Prove that $H^r_c(M; N)$ is isomorphic to $H^r_c(M \setminus N)$.

[1]Source: *Basic Homology*, 2nd. ed., by Elon L. Lima, IMPA 2012.