ELEMENTS OF SIMPLICIAL HOMOLOGY

1. Definitions.

(Open) p-simplex in R^m: (open) convex hull of $(p+1)$ affinely independent points $\{a_0, \dots, a_p\} \subset R^m$:

$$s = \{\sum_{i=0}^{p} t_i a_i, 0 < t_i < 1, \sum_i t_i = 1\} \subset R^m.$$

(The t_i are 'barycentric coordinates'.)

If s, t are simplices, $s < t$ (s is a face of t) if it is the (open) convex hull of a subset of the a_i (defining t).

Finite simplicial complex K in R^m: a finite collection of simplices (of various dimensions between 0 and $n = dim(K)$) satisfying:

(1) Every face of a simplex of K is a simplex of K.
(2) The (open) simplices of K are disjoint. Equivalently, two closed simplices in K are either disjoint, or intersect along a common face.

$K_0 \subset K$ is a *subcomplex* if it also satisfies (1) and (2).

The *polyhedron* $|K|$ of K is the compact subset of R^m: $|K| = \bigcup_{s \in K} s$.

It follows from the definition of simplicial complex that each subset of $Vert(X)$ that is the set of vertices of a simplex of K is the set of vertices of a *unique* simplex in K: each simplex in K is uniquely defined by its set of vertices.

A vertex mapping $\phi : Vert(K) \to Vert(L)$ defines a *simplicial map* ϕ from K to L if, whenever $a_0, \dots, a_p$ are vertices of a p-simplex in K, the points $\phi(a_0), \dots, \phi(a_p)$ are vertices of a simplex in L (which may be of lower dimension, if the vertex mapping is not injective.) We denote the corresponding mapping of simplices also by $\phi : K \to L$ For the corresponding polyhedra we have the continuous map:

$$|\phi| : |K| \to |L|, \quad \sum_i t_i a_i \mapsto \sum_i t_i \phi(a_i), \quad \sum_i t_i = 1.$$

The r-*dimensional chain group* of K is the free abelian group $C_r(K)$ with basis given by the r-dimensional simplices of K.

Consider now an *ordered* simplicial complex K: there exists a partial ordering of vertices inducing a total ordering of vertices in each simplex of K. We adopt the symbol $s_p = [a_0 a_1 \dots a_p]$ to denote an ordered p-simplex $s_p \in K$ (in particular the a_i are all distinct and affinely independent). We also set, for a permutation π of the vertices:

$$sgn(\pi)[a_0 \dots a_p] = [a_{\pi(0)} \dots a_{\pi(p)}].$$

Denote by $t_i = (-1)^i [a_o \dots \hat{a_i} \dots a_p]$, $i = 0, \dots, p$ the *ith* codimension 1 face of s_p, opposite the vertex a_i. This leads to the boundary operator:

$$\partial_r : C_p(K) \to C_{p-1}(K), \quad \partial s_p = \sum_{i=0}^{p} (-1)^i [a_0 \dots \hat{a_i} \dots a_p], \quad r > 0,$$

extended by linearity to an operator on $C_p(K)$. Set $\partial s_0 = 0$ if $s_0 \in C_0(K)$.

One shows easily that the composition $\partial\partial = 0$, thus $(C_r(K), \partial)_{r \geq 0}$ is a *chain complex*, and we have the subgroups $Z_r(K) = ker(\partial_r)$ of r-cycles and $B_r(K) = im(\partial_{r+1}) \subset Z_r(K)$ of r-boundaries, and the quotient group $H_r(K) = Z_r(K)/B_r(K)$, the r-th homology of K.

Next we want to associate a homomorphism of chain groups to a given simplicial map (suitable vertex map) $\phi : K \to L$ (say L is a simplicial complex in R^N.). This has the problem that for $s_p = [a_0 \dots a_p] \in C_p(K)$ (a simplex in K), the points $\phi(a_0) \dots \phi(a_p)$, although vertices of the same simplex in L, are not necessarily affinely independent in R^N, hence do not define an element of $C_p(L)$.

To get around this, we consider the f.g. abelian group with generators all symbols $\langle a_0 \dots a_p \rangle$ with $a_0, \dots, a_p$ vertices of a p-simplex in K (listed in any order), but allowing equalities among the a_i, and relations of two types:

$$\langle a_0 ... a_p \rangle = 0 \text{ if } a_i = a_j \text{ for some } i \neq j;$$

$$\langle a_{\pi(0)} \dots a_{\pi(p)} \rangle = sgn(\pi) \langle a_0 \dots a_p \rangle,$$

for any permutation π of $\{0, \dots, p\}$. The abelian group with these generators and relations is a free group, with basis:

$$\{\langle a_0 \dots a_p \rangle; [a_0 \dots a_p] \text{ a } p\text{-simplex of } K, a_0 < a_1 < \dots < a_p\}$$

(in the order of K), and is clearly isomorphic to $C_p(K)$ (hence we keep the same notation.)

It is easy to check that, defining ∂ by an analogous expression to the earlier one:

$$\partial \langle a_0 \dots a_p \rangle = \sum_i (-1)^i \langle a_0 \dots \hat{a_i} \dots a_p \rangle,$$

we have $\partial s = 0$ if s is degenerate (that is, includes repeated vertices). Thus $\partial : C_r(K) \to C_{r-1}(K)$ is well-defined by this formula (extended linearly to $C_r(K)$).

Now given $\phi : K \to L$ simplicial map, define $\phi_\# : C_r(K) \to C_r(L)$ (homomoprhism) via:

$$\phi_\#(\langle a_0 \dots a_p \rangle) = \langle \phi(a_0) \dots \phi(a_p) \rangle,$$

extended linearly to an operator $\phi_\# : C_p(K) \to C_p(L)$.

It is easy to check that $\phi_\#$ is zero on symbols including repetitions and that $\partial\phi_\# = \phi_\#\partial$ (ϕ is a chain map), and thus $\phi_\#$ on chains induces a homomorphism $\phi_* : H_p(K) \to H_p(L)$ in homology. We also have

$$(\phi\psi)_* = \phi_*\psi_*, \quad (id_K)_* = id_{H_p(K)}.$$

This is still a long way from being able to define an induced homomorphism associated to an arbitrary continuous map $f : |K| \to |L|$.

The following theorem is an easy observation.

Theorem. Let K be a finite ordered simplicial complex, $dim(K) = n$. Then:

(i) $H_q(K) = 0$ for $q > n$ (since $C_q(K) = 0$).

(ii) $H_r(K)$ is finitely generated, for $0 \leq r \leq n$.
(Since it is a quotient of the f.g. subgroup $Z_r(K)$ of the f.g. group $C_r(K)$, by the f.g. subgroup $B_r(K)$.)

(iii) $H_n(K)$ is a free group (possibly zero).
(Since it is the quotient of the free group $Z_n(K)$ by the trivial subgroup $\{0\}$; there are no n-boundaries.)

A subset $K_0 \subset K$ is a *subcomplex* of K if it is a simplicial complex. In particular, $Vert(K_0) \subset Vert(K)$.

Example. For $0 \leq r \leq dim(K)$, let K^r be the set of all simplices in K of dimension less than or equal to r, the 'r-skeleton' of K. This is a subcomplex, and K is a finite increasing union of its skeleta of increasing dimension.

Note the boundary operator restricts from the chain complex $C_*(K)$ to $C_*(K_0)$. Thus, defining the relative chain group as the quotient:

$$C_r(K, K_0) = C_r(K)/C_r(K_0),$$

the boundary operator $\partial : C_r(K, K_0) \to C_{r-1}(K, K_0)$ is well defined, and we have a relative chain complex $(C_r(K, K_0), \partial)_{r\geq 0}$. The homology of this chain complex is the relative homology of the pair (K, K_0):

$$H_r(K, K_0) = Z_r(K, K_0)/B_r(K, K_0).$$

Simplicial maps of pairs $\phi : (K, K_0) \to (L - L_0)$ are defined as before, leading to induced homomorphisms of chain groups and homology groups:

$$\phi_\# : C_r(K, K_0) \to C_r(L, L_0), \quad \phi_* : H_r(K, K_0) \to H_r(L, L_0).$$

Stars. Let K be a (finite, ordered) simplicial complex, $s \in K$ a simplex. The star of s is the set of all (open) simplices in K of which s is a face:

$$st(s) = \{t \in K; s < t.\}.$$

Clearly $st(s)$ *is not* a subcomplex of K. However, $K \setminus st(s)$ is a (finite) subcomplex. In particular its space $|K \setminus st(s)|$ is a compact subset of R^m, hence closed in $|K|$. Since $|K|$ is the disjoint union of the simplices of K, we have:

$$|K| = \bigcup_{s\in K} s = (\bigcup_{s\in K\setminus st(s)} s) \sqcup (\bigcup_{s\in st(s)} s) = |K \setminus st(s)| \sqcup |st(s)|.$$

We conclude $|st(s)|$ is *open* in $|K|$. In particular:

$$\{|st(a)|; a \in Vert(K)\} \text{ is an open cover of } |K|.$$

Remark. If $s = [a_0 \dots a_p]$ is a simplex (open) in K, we have:

$$s \subset |st(a_0)| \cap \dots \cap |st(a_p)|.$$

(We don't distinguish bwtween s and $|s| \subset R^m$ in the notation). Conversely, if $\{b_0, \dots, b_r\}$ is any set of vertices in K whose stars intersect:

$$(\exists x \in |K|) x \in |st(b_0)| \cap \dots \cap |st(b_r)|,$$

then each b_i is a vertex of s, the unique (open) simplex in K containing x (the 'carrier' of x.) We conclude:

A set of vertices of K spans a simplex of K iff their stars in K have a point in common.

Cones. Let K be a (finite, oriented) polyhedron, a a vertex of K. The set K_a of simplices of K not containing a is a subcomplex of K For $s \in K_a, s = \langle a_0 \dots a_r\rangle$, define $a * s = \langle aa_0 \dots a_r\rangle$, a simplex of dimension $r + 1$. (This works for $s \in K$ too, but the result is zero if a is a vertex of s.) We say K *is a cone of vertex* a if for all $s \in K_a$, $a * s$ is a simplex in K.

Then if K is a cone with vertex a, we have:

$$K = \bigcup_{s\in K_a} a * s.$$

(We call K_a the 'base' of the cone, of course.)

Let K be a cone with vertex a. By linear extension, we obtain a map $a* : C_r(K) \to C_{r+1}(K)$. Using the definition of ∂, we check directly that:

$$\partial(a * s) = s - a * \partial s \quad \forall s \in K, \text{ and thus } \partial(a * x) = x - a * \partial x, x \in C_r(K).$$

Thus if $z \in Z_r(K)$ for $r \geq 1$ we have z is a boundary, and we see that $H_r(K) = 0, r \geq 1$ (if K is a cone).

2. Barycentric subdivision and simplicial approximation.

Given a continuous map of polyhedra $f : |K| \to |L|$, ideally it should be possible to define an induced homomorphism in homology. We know how to do this for simplicial maps, so as a first step we need to approximate a general map by a simplicial one. This involves the important idea of 'barycentric subdivision'. From K we obtain a new simplicial complex K' with $|K| = |K'|$–the polyhedron is the same, but triangulated into smaller pieces.

Let K be a finite oriented simplicial complex in R^m.For any simplex $= \langle a_0 \dots a_p \rangle$ in K, let $b_s = \frac{1}{p+1} \sum_{i=0}^{p} a_i \in R^m$ be the barycenter of s. K' is defined inductively, as follows. For the 0-skeleton $K^0 = Vert(K)$, we set $(K^0)' = K^0$, no change. Assume the barycentric subdivision of the r-skeleton K^r is defined. Now let s be a simplex in K, of dimension $r + 1$. Denote by $bd(s)$ the set of its proper faces (a subcomplex of K^r, hence $(bd(s))'$ is defined, but is not a subcomplex of K^r). Then set:

$$s' = b_s * (bd(s))',$$

the cone over the boundary subcomplex, with vertex the barycenter of s. (This is a simplicial complex, but *is not* a subcomplex of s.) Then define:

$$(K^{r+1})' = \bigcup_{s \in K^{r+1}} s',$$

a new simplicial complex. Finally, set $K' = (K^n)'$, where $n = dim(K)$. We have:

$$Vert(K') = \{b_s; s \in K\} \quad \text{and } K' = \{\langle b_{s_0} \dots b_{s_p} \rangle, s_i \in K, s_0 < s_1 < \dots < s_p, 0 \leq p \leq dim(K)\}$$

(The chain of inequalities denotes 'successive proper faces'.)

Note $|K'| = |K|$, since $|s'| = |s|$ for any simplex $s \in K$.

Next we wish to define a 'subdivision chain map' $Sd : C_r(K) \to C_r(K')$, and we do so by induction on dimension, setting Sd to be the identity on $C_0(K) \subset C_0(K')$. Assume $Sd : C_{r-1}(K) \to C_{r-1}(K')$ has been defined (and is a chain map), and let $s \in K$ be an r-simplex. Set:

$$Sd(s) = b_s * Sd(\partial s),$$

noting $\partial s \in C_{r-1}(K), Sd(\partial s) \in C_{r-1}(K')$, and using the map b_s* from $C_{r-1}(K')$ to $C_r(K')$ defined in the previous section. By induction, we assume

$$\partial(Sd(x)) = Sd(\partial x), \quad x \in C_p(K), 1 \leq p \leq r - 1.$$

Now recall from Section 1: $\partial(a * x) = x - a * \partial x$, and let $x = \sum n_i s_i \in C_r(K)$ (finite sum, the s_i are r-simplices in K and $n_i \in \mathbb{Z}$.) We compute:

$$\partial(Sd(x)) = \sum_i n_i \partial(Sd(s_i)) = \sum_i n_i \partial(b_{s_i} * Sd(\partial s_i))$$

$$= \sum_i n_i (Sd(\partial s_i) - b_{s_i} * \partial(Sd(\partial s_i))$$

$$= Sd(\partial x) - \sum_i n_i b_{s_i} * Sd(\partial\partial s_i) \quad \text{(using the induction hypothesis)},$$

which clearly equals $Sd(\partial x)$. Thus Sd is a chain map.

Reduction of diameter. The diameter of a simplex is the max distance between its vertices. We *claim* that if $dim(s) = n$, the diameter of each simplex in s' (the subdivision of s) is less than or equal to $\frac{n}{n+1} diam(s)$.

Proof. We use induction on the dimension n of s. Consider $|w_j - w_k|$, the distance between two vertices of $[w_0 \dots w_n]$, simplex of s'. If both w_j, w_k are in a proper face of s, we're done by induction (note $\frac{n-1}{n} < \frac{n}{n+1}$.) Thus we may assume $w_j = b, , w_k = v_i$, the barycenter and a vertex of s. We have:

$$b_i = \frac{1}{n} \sum_{j \neq i} v_j, \quad b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i,$$

where b_i is the barycenter of the face $[v_0 \dots \hat{v_i} \dots v_n]$ of s, opposite to v_i. So:

$$|b - v_i| = |\frac{n}{n+1} b - \frac{n}{n+1} v_i| = \frac{n}{n+1} |b_i - v_i| \leq \frac{n}{n+1} diam(s),$$

as claimed.

For a (finite) simplicial complex K, define the *mesh* $\mu(K)$ as the max diameter of a simplex in K. Then $\mu(K') \leq \frac{n}{n+1} \mu(K)$ if $dim(K) = n$. Denote by $K^{(j)}$ the j-fold iterated subdivision of K. Clearly $\mu(K^{(j)}) \to 0$ as $j \to \infty$.

Simplicial approximation. K, L finite simplicial complexes, $f : |K| \to |L|$. $\phi : K \to L$ (simplicial map) is a simplicial approximatiom of f if:

$$\forall x \in |K|, a \text{ vertex of carrier}(x) \Rightarrow \phi(a) \text{ vertex of carrier } (f(x)).$$

(Recall the 'carrier' of a point in $|K|$ is the unique open simplex of K containing x.)

Equivalently: for any vertex a of K, $f(|st(a)|) \subset |st(\phi(a))|$.

Simplicial Approximation Theorem. Given $f : |K| \to |L|$ continuous, there exists an n and $\phi : K^{(n)} \to L$ simplicial approximation of f.

Proof. Let $\delta > 0$ be a Lebesgue number of the open cover of $|L|$ given by the stars of its vertices, By uniform continuity of f, there exists $\epsilon > 0$ so that any subset of $|K|$ with diameter less than ϵ has image with diameter less than δ, hence contained in the star of some vertex of L. Choose n so that all simplices of $K^{(n)}$ have diameter less than $\epsilon/2$. Then for any a, vertex of $K^{(n)}$, $diam(|st(a)|) < \epsilon$, so there is a vertex $\phi(a)$ of L so that $f(|st(a)|) \subset |st(\phi(a))|$. ϕ is a simplicial approximation of f.

One consequence is that we can define a homomorphism in homology associated to a continuous map $f : |K| \to |L|$. Let $\phi : K^{(n)} \to L$ bas a simplicial approximation, and consider $Sd^n_{\#} : C(K) \to C(K^{(n)})$ the homomorphism of chain groups defined by iterated subdivision. Then set:

$$f_* = \phi_* \circ (Sd^n)_* = (\phi \circ Sd^n)_* : H_r(K) \to H_r(L).$$

Problem: We need to show this map is independent of n, and independent of the approximation ϕ, for given n. This can be done, but is quite technical . It involves the following combinatorial analog of homotopy:

Definition. Two simplicial maps $\phi, \psi : K \to L$ are *contiguous* if for any simplex $s \in K$ there exists a simplex $t \in L$ so that both $\phi(s)$ and $\psi(s)$ are faces of t.

Remark: Contiguity is not a transitive relation. (Can you think of an example?)

For continuous maps to a polyhedron, we have the *definition:* $f, g : X \to |L|$ are *approximate* if for any $x \in X$, there exists $t(x) \in L$ so that $f(x), g(x)$ are both in the closure of $t(x)$.

(i) Two approximate maps are homotopic. Say $|L| \subset R^m$, and define $F : X \times I \to R^m$ by $F(t,s) = (1-s)f(x) + sg(x)$. Note $F(x,s) \in \overline{t(x)}$, so F maps into $|L|$.

(ii) If ϕ is a simplicial approximation of f, for $x \in |K|$ we have $|\phi|(x)$ is in the closure of the carrier of $f(x)$, so $|\phi|$ and f are approximate, hence homotopic.

(iii) If ϕ, ψ are contiguous simplicial maps from K to L, $|\phi|, |\psi|$ are approximate, hence homotopic.

One may show that any two simplicial approximations ϕ (defined on $K^{(r)}$) and ψ (defined on $K^{(s)}$) to the same $f : |K| \to |L|$ are contiguous, hence homotopic. [HW 1.7.11].

Example 1. Let K be the simplicial complex defined by the (oriented) faces of an $(n+1)$ simplex s, K^n its n-skeleton. One sees easily that $|K^n|$ is homeomorphic to S^n, hence gives a triangulation of S^n. We compute the homology of this triangulation.

Since K is a cone, its homology is trivial in dimensions greater than 0. This easily implies $H_r(K^n) = 0$ for $0 < r < n$. Let $w \in C_n(K^n)$ be the sum of all oriented n-dimensional faces of s. Then $w = \partial s$ (in K), so in fact $w \in Z_n(K^n)$; since there are no $(n+1)$-simplices in K^n, $[w] \neq 0$ in $H_n(K^n)$. In fact $H_n(K^n) \approx \mathbb{Z}$, and $[w]$ is a generator: if $z \in Z_n(K^n)$ is a cycle, we have $z = \partial x$ for some $x \in C_{n+1}(K)$; and s is the only $(n+1)$-simplex in K, so $x = Ns$ for some $N \in \mathbb{Z}$. Thus $z = \partial x = N\partial s = Nw$ in $Z_n(K^n)$, and $[z] = N[w]$ in $H_n(K^n)$.

Example 2. Euler characteristic of polyhedra. (See [Rotman] p. 145-146.)