PROBLEMS ON SIMPLICIAL HOMOLOGY

1. [Rotman p. 87] (i) Consider a short exact sequence of finitely generated abelian groups:

$$0 \to A \to B \to C \to 0.$$

Prove that $rank(B) = rank(A) + rank(C)$.

(*Hint:* Extend a maximal independent subset of A to a maximal independent subset of B.)

(ii) Now consider the exact sequence of finitely generated abelian groups:

$$0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to 0.$$

Prove that:

$$\sum_{i=0}^{n} (-1)^i rank(A_i) = 0.$$

2. (i) Let X be the figure-eight space. Exhibit a triangulation K of X (that is, a simplicial complex in R^2 with $|K|$ homeomorphic to X.) Pick bases for the chain groups $C_0(K), C_1(K)$ and compute the kernel and image of the boundary operator $\partial : C_1(K) \to C_0(K)$ with respect to those bases. Then find $H_1(K)$.

(ii) Generalize the result to the space 'bouquet of r circles meeting at a point x_0'.

3. Given a finite simplicial complex K in R^n, regard $R^n = R^n \times 0 \subset R^{n+1}$ and define the *suspension* of K as

$$SK = S^+K \sqcup S^-K, \quad S^+K = e_{n+1} * K, \quad S^-K = -e_{n+1} * K.$$

(i)Show that SK is a simplicial complex.

(ii) Prove that $H_{r+1}(SK) \approx H_r(K)$, for all $r > 0$. (*Hint:* Use the Mayer-Vietoris sequence in simplicial homology.)

4. [Rotman p. 153]. Recall that if $(X, x_0), (Y, y_0)$ are pointed spaces, the *wedge* $X \vee Y$ is the quotient space of their disjoint union $X \sqcup Y$ in which the basepoints are identified.

(i) If K_1, K_2 are simplicial complexes, show that so is $K_1 \vee K_2$, and that for all $n \geq 1$:

$$H_n(K_1 \vee K_2) \approx H_n(K_1) \oplus H_n(K_2).$$

(*Hint:* Mayer-Vietoris sequence in simplicial homology.)

(ii) Let $m_1, \ldots m_n$ be a sequence of nonnegative integers. Prove there exists a connected simplicial complex of dimension n with $H_q(K)$ free abelian of rank m_q, for each $q = 1, \ldots, n$.