SUFFICIENCY OF SMOOTH SINGULAR CHAINS

For a smooth manifold M, denote by $\{S_r^\infty(M), \partial\}_{r\geq 0}$ the chain complex of smooth singular chains. This is generated (with $\mathbb{Z}$ coefficients) by smooth singular simplices $\sigma : \Delta_r \to M$, where 'smooth' means the restriction to Δ_r of a smooth map $U \to M$, defined in an open neighborhood $U \subset \mathbb{R}^r$ of Δ_r. Denote by $\{H_r^\infty(M)\}_{r\geq 0}$ the homology of this chain complex.

We have an inclusion map $i : S_r^\infty(M) \to S_r(M)$, clearly a chain map (from the definition of ∂.)

Theorem. The induced map $i_* : H_r^\infty(M) \to H_r(M)$ to singular (continuous) homology groups is an isomorphism.

Proof. We may assume $M \subset \mathbb{R}^N$; let $V \subset \mathbb{R}^N$ (open) be a tubular neighborhood of M, $\pi : V \to M$ the smooth closest-point projection. Consider the open cover of M:

$$\mathcal{O} = \{B \cap M; B \subset \mathbb{R}^N \text{ open euclidean ball contained in } V\}.$$

Denote by $S_r^{\mathcal{O}}(M)$ the complex of $\mathcal{O}$-small (continuous) singular chains of M, with $\mathbb{Z}$ coefficients. Note that if $\sigma : \Delta_r \to M$ is continuous and $\mathcal{O}$-small, the line joining two vertices $\sigma(e_i), \sigma(e_j)$ is contained in V. Thus V contains the affine simplex:

$$\sigma' = \langle \sigma(e_0), \sigma(e_1), \ldots, \sigma(e_r)\rangle.$$

(Recall that we allow degenerate simplices, or orientation reversal with a sign change.) In addition, σ' is parametrized by the map from Δ_r:

$$\text{With } t_i \in [0,1] \text{ and } \sum_{i=1}^r t_i \leq 1 : \sum_{i=1}^r t_i e_i \mapsto \sum_{i=1}^r t_i \sigma(e_i),$$

which is clearly the restriction of a smooth (indeed linear) map from a neighborhood of Δ_r in $\mathbb{R}^r$. Thus σ' is a smooth simplex in V. Composing with $\pi : V \to M$ we obtain a smooth singular r-simplex $\bar{\sigma} : \Delta_r \to M$. The assignment $\sigma \mapsto \bar{\sigma}$ is clearly compatible with the boundary operator, hence defines a chain map:

$$\varphi : S_r^{\mathcal{O}}(M) \to S_r(M), \quad \varphi(\sigma) = \bar{\sigma},$$

(extended linearly with $\mathbb{Z}$ coefficients) inducing a homomorphism φ_* of homology groups.

Claim: $\varphi_* : H_r^{\mathcal{O}}(M) \to H_r^\infty(M)$ (smooth singular homology with $\mathbb{Z}$ coefficients) is an isomorphism, with inverse i_*. (Recall $H_r^{\mathcal{O}}(M) \approx H_r(M)$.)

Proof of claim. The main idea is to observe the assignment $\sigma \mapsto \bar{\sigma}$ is the end result of a deformation. Namely, given $\sigma : \Delta_r \to M$ ($\mathcal{O}$-small), consider the deformation:

$$P_\sigma : \Delta_r \times [0,1] \to M$$

defined as the straight-line homotopy (in some $B \subset V$) from σ to σ', followed by projection to M via π. We have:

1) For $x \in \Delta_r$, $P_\sigma(x,0) = \sigma(x)$, $\quad P_\sigma(x,1) = \bar{\sigma}(x)$.

2) $\partial_i P_\sigma = P_{\partial_i \sigma}$, where $\partial_i P_\sigma = P_{\sigma|(\partial_i \Delta_r)\times[0,1]}$.

From this point on, the proof follows the lines of the proof of homotopy invariance of singular homology. In P_σ, let $v_i = (e_i, 0), w_i = (e_i, 1)$ and using the simplicial decomposition:

$$\Delta_r \times [0,1] = \bigcup_{i=0}^{r} [v_0, \ldots, v_i, w_i, \ldots, w_r] \subset \mathbb{R}^r \times \mathbb{R}$$

define the 'prism operator' $D : S_r^{\mathcal{O}}(M) \to S_{r+1}^{\mathcal{O}}(M)$ by:

$$D\sigma = \sum_{i=0}^{r} (-1)^i P_{\sigma|[v_0,\ldots,v_i,w_i,\ldots w_r]},$$

extended by $\mathbb{Z}$-linearity.

We have:

$$\partial D\sigma + D\partial\sigma = \sigma - \bar{\sigma}.$$

(This is proved exactly as in the proof of homotopy invariance [Hatcher, p. 112].) Thus any r-cycle $z \in S_r^{\mathcal{O}}(M)$ is homologous to the differentiable cycle $\bar{z} = \varphi(z)$. This implies, in homology: $i_*\varphi_*[z] = [z]$, or $i_*\varphi_* = id$ in $H_r^{\mathcal{O}}(M)$ (which may be identified with $H_r(M)$.)

Note $\varphi \circ i$ is not the identity on $S_r^\infty(M)$, so we need to check that $\varphi_* i_*$ is the identity in $H_r^\infty(M)$. If σ is a smooth singular r-simplex, we may assume the deformation P_σ is smooth (in a neighborhood of $\Delta_r \times [0,1]$ in $\mathbb{R}^{r+1}$). So the prism operator D restricts to a homomorphism $D : S_r^\infty(M) \to S_{r+1}^\infty(M)$. Hence any smooth r-cycle z is homologous in $S_r^\infty(M)$ to $\bar{z} = \varphi(z)$, or $\varphi_* i_*[z] = [z]$.

The situation in cohomology is similar, as follows from duality. With $S_\infty^r = Hom(S_r^\infty, \mathbb{Z})$ and $S_{\mathcal{O}}^r = Hom(S_r^{\mathcal{O}}, \mathbb{Z})$, we have $i^T : S_{\mathcal{O}}^r(M) \to S_\infty^r(M)$, $\varphi^T : S_\infty^r(M) \to S_{\mathcal{O}}^r(M)$ commuting with the codifferentials δ, and:

$$\varphi^{T*} : H_\infty^r(M) \to H_{\mathcal{O}}^r(M) \approx H^r(M)$$

is an isomorphism with inverse i^*. ($i^*[u]$ is the cohomology class of the restriction of the cocycle u to smooth $\mathcal{O}$-small r-chains.)

This follows by dualizing the preceding argument, in particular the transpose prism operator $D^T : S_{\mathcal{O}}^{r+1}(M) \to S_{\mathcal{O}}^r(M)$, which satisfies on $S_{\mathcal{O}}^r(M)$:

$$\delta D^T + D^T \delta = u - \varphi^T u.$$