NOTES ON THE EXCISION THEOREM

1. From sufficiency of 'split chains' to excision.

We consider the excision theorem in homology in one of two settings:

(1) simplicial homology: X_1, X_2 are subcomplexes of a simplicial complex X, with $X = X_1 \cup X_2$;

(2) singular homology: X_1, X_2 are subspaces of a space X, with $X = int(X_1) \cup int(X_2)$

In both cases the excision theorem is the statement that the inclusion of pairs $i : (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces an isomorphism of the corresponding relative homology groups of any dimension $n \geq 0$.

A key step in the proof is the claim that 'split chains suffice' to compute the homology of the combined space X. Namely, consider the inclusion of chain groups (singular or simplicial, depending on the case):

$$j : C_n(X_1 + X_2) \hookrightarrow C_n(X)$$

The complex on the left denotes 'split chains' $\gamma = \gamma_1 + \gamma_2$, sums of n-chains with values in X_1 and n-chains with values in X_2. In the simplicial case, this is an equality, since any simplex in the simplicial complex X is either in the subcomplex X_1, or in X_2 (or both), *unlike* a general singular simplex in a space $X = X_1 \cup X_2$ (in the singular homology setting.) By 'split chains suffice' we mean either of the two equivalent statements:

(i) The inclusion of chain groups induces an isomorphism $j_* : H_n(X_1+X_2) \to H_n(X)$. (The group on the left is the homology of the complex $C_*(X_1 + X_2)$ of 'split chains');

(ii) The relative homology groups $H_n(X, X_1 + X_2)$ vanish. These are the homology groups of the relative complex

$$C_n(X, X_1 + X_2) := C_n(X)/C_n(X_1 + X_2).$$

Problem 1. Show that (i) and (ii) are equivalent. That (ii)$\Rightarrow$ (i) follows from considering the long exact sequence corresponding to:

$$0 \to C_n(X_1 + X_2) \to C_n(X) \to C_n(X, X_1 + X_2) \to 0.$$

Now prove (i)$\Rightarrow$ (ii).

We interpret 'split chains suffice' geometrically as the following two statements, where in (1) we use condition (ii) and in (2) we use condition (i).

(1) (surjectivity in homology): For any $z \in C_n(X)$ with $\partial z \in C_{n-1}(X_1+X_2)$, we have: $z \sim z_1 + z_2$ for some $z_i \in C_n(X_i)$ (meaning $z = z_1 + z_2 + \partial w, w \in C_{n+1}(X)$).

(2) (injectivity in homology): $z_1 + z_2 \sim 0$ in X (where $z_i \in C_n(X_i)$ and $\partial(z_1 + z_2) = 0) \Rightarrow z_1 + z_2 = \partial(w_1 + w_2), w_i \in C_{n+1}(X_i)$.

We then have the following:

Lemma. If (1) and (2) hold, then excision holds for (X, X_1, X_2), as above: the inclusion $(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces isomorphisms:

$$H_n(X_1, X_1 \cap X_2) \approx H_n(X, X_2), \quad n \geq 0.$$

Remark: In particular, the lemma implies excision holds in the simplicial case, where 'split chains suffice' is trivial.

Proof. (a) The homomorphism induced in homology by inclusion is *surjective:* let $\alpha \in C_n(X)$ be a relative cycle in (X, X_2): $\partial\alpha \in C_{n-1}(X_2)$. By (1) above, $\alpha = \alpha_1 + \alpha_2 + \partial w$, where $\alpha_i \in C_n(X_i)$. Thus $\partial\alpha = \partial\alpha_1 + \partial\alpha_2$, which shows $\partial\alpha_1 \in C_{n-1}(X_1 \cap X_2)$. Hence $\alpha \sim \alpha_1$ in (X, X_2) (that is, α is homologous in X mod X_2 to the image under inclusion of α_1, a relative $(X_1, X_1 \cap X_2)$ cycle.)

(b) The homomorphism induced in homology by inclusion is *injective*: let $z_1 \in C_n(X_1)$ be a relative cycle: $\partial z_1 \in C_{n-1}(X_1 \cap X_2)$. Suppose $z_1 \sim 0$ in (X, X_2): $z_1 = \partial w + z_2$, for some $w \in C_{n+1}(X), z_2 \in C_n(X_2)$. Then $z_1 - z_2 \sim 0$ in X. By (2), in fact $z_1 - z_2 = \partial(w_1 + w_2)$, where $w_i \in C_{n+1}(X_i)$. But then $z_1 = \partial w_1 + (z_2 + \partial w_2)$, showing, first, that $z_2 + \partial w_2 \in C_n(X_1 \cap X_2)$; and second, that $z_1 \sim 0$ in $(X_1, X_1 \cap X_2)$.

Remark. Cp. the algebraic arguments in [Rotman, Lemma 6.11] and Hatcher, p. 124]. The converse of the statement in the lemma is not true.

2. From excision to the Mayer-Vietoris sequence.

Still in the situation $X = int(X_1) \cup int(X_2)$ (or X, a simplicial complex, is the union of two subcomplexes X_1 and X_2), consider the short exact sequence of chain groups:

$$0 \to C_n(X_1 \cap X_2) \xrightarrow{i_*} C_n(X_1) \oplus C_n(X_2) \xrightarrow{j_*} C_n(X_1 + X_2) \to 0,$$

where $i(w) = (w, w)$ and $j(w_1, w_2) = w_1 - w_2$.

From 'split chains suffice', we have $H_n(X_1 + X_2) \approx H_n(X)$. This leads to the following long exact sequence:

$$\cdots \to H_n(X_1 \cap X_2) \xrightarrow{i_*} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(X_1 \cap X_2) \to \ldots$$

Here the meanings of i_*, j_* are clear. The definition of the 'connecting operator' ∂_* is the following: given $[z] \in H_n(X)$, by 'split chains suffice' (statement (1)) we may assume $z = z_1 - z_2$, where $z_i \in C_n(X_i)$ with $\partial z_1 = \partial z_2 \in C_{n-1}(X_1 \cap X_2)$. Then let $\partial_*[z] = [\partial z_1] \in H_{n-1}(X_1 \cap X_2)$.

Problem 2: (i) Show this is well-defined (that is, independent from the choices of z, z_1, z_2.)

(ii) Prove directly this sequence is exact, at each of the three steps. That is, supply proofs of each of the three equalities ($ker(j_*) = im(i_*)$, etc.) in question, using only the definition of the maps in homology.

Observation/problem 3: As a consequence, if $X_1 \cap X_2$ is contractible (in particular, path-connected) we have $H_n(X) \approx H_n(X_1) \oplus H_n(X_2)$, if $n \geq 2$. What happens when $n = 1$? Is this still true? (Proof or counterexample.)

3. Main steps in the proof of 'split chains suffice'.

(Outline, based on [Rotman, ch.6] and [Hatcher, p. 119-124].) We now use S_n for the chain complexes (since the proof is in singular homology.)

Step 1. Define a 'subdivision operator' $\mathcal{S}_n^X : S_n(X) \to \mathcal{S}_n(X)$, and show it is a chain map.

As a notational matter, let $\delta_n : \Delta_n \to [e_0 \dots e_n]$ be the identity (we need to parametrize Δ_n itself as a 'singular simplex' in $\mathbb{R}^n$.)

This is done in two parts. First, assuming $X = E$ is a 'convex space' (convex subset of some euclidean space; say, an affine subspace of euclidean space, or a simplex.) Then for an *affine* 'singular simplex' $\tau : \Delta_n \to E$, we define its subdivision, inductively on n, via the 'cone construction' from $\tau(b_n)$, the image under τ of the barycenter b_n of Δ_n ($n \geq 1$):

$$\mathcal{S}_n^E \tau := \tau(b_n) \cdot \mathcal{S}_{n-1}^E(\partial_n \tau).$$

Extend linearly to obtain $\mathcal{S}_n^E : S_n(E) \to S_n(E)$.

Now establish the chain map property for the $\mathcal{S}_n^E$ by induction on n. Using the fact the coning map $b\cdot : S_n(E) \to S_{n+1}(E)$ is a chain homotopy from the zero map to the identity (meaning $\partial(b \cdot \gamma) + b \cdot (\partial \gamma) = \gamma$), one easily finds:

$$\partial(\mathcal{S}_n^E \tau) = \mathcal{S}_{n-1}^E(\partial \tau) - \tau(b_n) \cdot \partial(\mathcal{S}_{n-1}^E \partial \tau) = \mathcal{S}_{n-1}^E(\partial \tau),$$

using the induction hypothesis at the second step.

Now let X be a general space. Given a singular simplex $\sigma : \Delta_n \to X$, regarding Δ_n as the 'singular simplex' $\delta_n \in S_n(E)$, $\delta_n : \Delta_n \to E$ (where $E = \Delta_n$), we define, using the chain map $\sigma_\# : S_n(E) \to S_n(X)$:

$$\mathcal{S}_n^X(\sigma) := \sigma_\# \mathcal{S}_n^E(\delta_n).$$

Extend linearly to obtain $\mathcal{S}_n^X : S_n(X) \to S_n(X)$. It is then an easy formal matter to derive the chain map property for $\mathcal{S}_n^X$ from that of $\mathcal{S}_n^E$ (using also the fact $\sigma_\#(\delta_n) = \sigma \in S_n(X)$).

Step 2. Construction of a map $T_n : S_n(X) \to S_{n+1}(X)$, chain homotopy from $\mathcal{S}_n$ to the identity on $S_n(X)$:

$$\partial T_n + T_{n-1}\partial = Id_{S_n(X)} - \mathcal{S}_n.$$

As in Step 1, first we do this in the case $X = E$, a 'convex space". One easily shows (assuming, inductively, the chain homotopy property holds for T_{n-1}) that if $\gamma \in S_n(E)$, the chain $\gamma - \mathcal{S}_n\gamma - T_{n-1}^E \partial \gamma$ is a *cycle.* As recalled in step 1, on cycles φ in E the cone operation is a 'one-sided inverse' to ∂, in the sense $\partial(b \cdot \varphi) = \varphi$. Fix $b \in E$ arbitrarily, and define, inductively, for $\gamma \in S_n(E)$:

$$T_n^E \gamma = b \cdot (\gamma - \mathcal{S}_n \gamma - T_{n-1}^E \partial \gamma) \in S_{n+1}(E).$$

The chain homotopy property for T_n^E follows directly from this definition:

$$\partial(T_n^E \gamma) = \gamma - \mathcal{S}_n \gamma - T_{n-1}^E \partial \gamma.$$

As in Step 1, we transfer this construction to a general space X by first defining T_n^X for a singular simplex $\sigma : \Delta_n \to X$. For $E = \Delta_n$ and $\sigma_\# : S_n(E) \to S_n(X)$:

$$T_n^X(\sigma) = \sigma_\# T_n^E \delta_n \in S_{n+1}(X),$$

where $\delta_n \in S_n(E)$ is the identity. Then extend by linearity to $T_n^X : S_n(X) \to S_{n+1}(X)$. It is then easy to show T_n^X is a chain homotopy from $\mathcal{S}_n$ to the identity in $S_n(X)$ (using also that the subdivision operator $\mathcal{S}_n$ is natural with respect to maps $f : X \to Y$.)

As an important *corollary* of Steps 1 and 2, we have that if $z \in Z_n(X)$ is a cycle, then so is $\mathcal{S}_n z$, and $z \sim \mathcal{S}_n z$ in $H_n(X)$. In other words, $\mathcal{S}_n$ induces the identity map on $H_n(X)$.

Step 3. Suppose $X = int(X_1) \cup int(X_2)$, let $\gamma \in S_n(X)$ be a chain. Then there exists an integer $q = q(\gamma) \geq 1$ so that, for the iterated subdivision $\mathcal{S}_n^q \gamma$, we have:

$$\mathcal{S}_n^q \gamma \in S_n(X_1 + X_2).$$

Idea of proof. For chains in a convex space E, we know $mesh(\mathcal{S}_n^q \gamma) \to 0$ as $q \to \infty$. Then the claim follows from a Lebesgue number argument, for the open cover $\{int(X_1), int(X_2)\}$ of the compact set defined by the image of γ in X.

Step 4. Theorem. ('split chains suffice'). The operator

$$\Theta_n : H_n(X_1 + X_2) \to H_n(X)$$

induced by the inclusions $S_n(X_1 + X_2) \hookrightarrow S_n(X)$ is an isomorphism.

Proof. Surjectivity is easy. Let $[z] \in H_n(X)$, $\partial z = 0$. Then for some $q = q(z)$, we may write $\mathcal{S}_n^q z = z_1 + z_2$, for some $z_i \in S_n(X_i)$, Since $\mathcal{S}^q$ is a chain map, we have $\partial(z_1 + z_2) = 0$. Then:

$$\Theta_n [z_1 + z_2]_{H_n(X_1 + X_2)} = [z_1 + z_2]_{H_n(X)} = [\mathcal{S}_n^q z]_{H_n(X)} = [z]_{H_n(X)}.$$

The proof of injectivity includes a subtle point. (See [Rotman, p. 117-118], corrected below.) Suppose $[\gamma_1 + \gamma_2]_{H_n(X)} = 0$: $\exists \beta \in S_{n+1}(X), \partial \beta = \gamma_1 + \gamma_2$. Then for some $q = q(\beta), \mathcal{S}_{n+1}^q \beta = \beta_1 + \beta_2, \beta_i \in S_{n+1}(X_i)$. Since $\mathcal{S}^q$ is a chain map, we have $\partial(\beta_1 + \beta_2) = \mathcal{S}_n^q(\gamma_1 + \gamma_2)$. This shows $[\mathcal{S}_n^q(\gamma_1 + \gamma_2)]_{H_n(X_1 + X_2)} = 0$.

Issue: Although we know $\mathcal{S}_n^q$ acts as the identity in $H_n(X)$, we have not established this for $H_n(X_1 + X_2)$.

To resolve this, note that the subdivision operator $\mathcal{S}_n$ maps $S_n(X_i)$ to $S_n(X_i)$, and thus maps $S_n(X_1+X_2)$ to itself (i.e., subdivision preserves the space of 'split chains'.) Likewise, the chain homotopy T_n restricts to $T_n' : S_n(X_1) \to S_{n+1}(X_1), T_n'' : S_n(X_2) \to S_{n+1}(X_2)$. Now consider iterated subdivision (dropping dimension subscripts from the notation from now on). Define for $q \geq 0$:

$$D_q : S_n(X) \to S_{n+1}(X), \quad D_q = \sum_{0 \leq i < q} T\mathcal{S}^i.$$

A standard calculation gives (cf. [Hatcher, p.123]):

$$\partial D_q + D_q \partial = Id - \mathcal{S}^q \text{ on } S_n(X);$$

D_q is a chain homotopy operator from $\mathcal{S}^q$ to the identity. We also have restrictions of D_q to the chain complexes of X_1 and X_2:

$$D_q' = \sum_{0 \leq i < m} T'\mathcal{S}^i, \quad D_q'' = \sum_{0 \leq i < m} T''\mathcal{S}^i.$$

Getting back to the proof of injectivity, we may write:

$$\gamma_1 - \mathcal{S}^q \gamma_1 = (D_q'' \partial + \partial D_q')\gamma_1, \quad \gamma_2 - \mathcal{S}^q \gamma_2 = (D_q'' \partial + \partial D_q')\gamma_2.$$

Adding the two, we find:

$$\gamma_1 + \gamma_2 - \mathcal{S}^q(\gamma_1 + \gamma_2) = (D_q' \partial \gamma_1 + D_q'' \partial \gamma_2) + \partial(D_q' \gamma_1 + D_q'' \gamma_2),$$

where the first term on the right equals $D_q \partial(\gamma_1 + \gamma_2) = 0$. This shows:

$$[\gamma_1 + \gamma_2]_{H_n(X_1+X_2)} = [\mathcal{S}^q(\gamma_1 + \gamma_2)]_{H_n(X_1+X_2)} = 0,$$

establishing the injectivity claim.