UNIVERSAL COEFFICIENTS IN COHOMOLOGY AND $Ext(H;G)$.

Consider a chain complex $\{C_*, \partial\}$ of free abelian groups:

$$\dots C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \dots$$

as well as the dual complex $\{C^*, \delta\}$, with $C^n = Hom(C_n; G)$ for a fixed abelian group G, and $\delta = \partial^T$, i.e. for the Kronecker pairing $\langle u, x\rangle = u(x)$, we define: $\langle \delta u, x\rangle = \langle u, \partial x\rangle$:

$$\dots \to C^{n-1} \xrightarrow{\delta} C^n \xrightarrow{\delta} C^{n+1} \to \dots$$

Our goal is to show the cohomology groups $H^n(C;G)$ of C^* are completely determined by G and the homology groups $H_n(C)$

1. There is a natural homomorphism:

$$h : H^n(C;G) \to Hom(H_n(C);G),$$

defined as follows: a class in $H^n(C;G)$ is represented by a hom. $u : C_n \to G$ s.t. $\delta u = 0$, i.e $u\partial = 0$, i.e. $u = 0$ on B_n. Thus the restriction $u_0 = u_{|Z_n}$. induces $\overline{u_0} : Z_n/B_n \to G$, i.e. an element $\bar{u_0} \in Hom(H_n(C);G)$.

If $u \in Im(\delta)$, say $u = \delta v = v\partial$, then $u = 0$ on Z_n, thus $u_0 = 0$ and $\overline{u_0} = 0$. Thus:

$$h : H^n(C;G) \to Hom(H_n(C);G) \quad [u] \mapsto \overline{u_0}$$

is a well-defined homomorphism.

2. h is surjective. The short exact sequence:

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0$$

splits, since B_{n-1} is free, as a subgroup of the free abelian group C_{n-1}. Thus there exists a hom $p : C_n \to Z_n$ such that $p_{|Z_n} = id_{Z_n}$ (review [Hatcher, p.147].) So any hom $u_0 : Z_n \to G$ extends to $h = u_0 p : C_n \to G$. This extends homs. $Z_n \to G$ vanishing on B_n to homs. $C_n \to G$ vanishing on B_n, i.e. extends homs. $H_n \to G$ to elements of $Z^n(C;G)$. Thus we have a hom $Hom(H_n(C);G) \to Z^n$. Compose with the quotient hom. to get a hom $k : Hom(H_n(C);G) \to H^n(C;G)$, such that $h \circ k = id$ (identity in $Hom(H_n(C);G)$.) This shows the sequence:

$$0 \to Ker(h) \to H^n(C;G) \xrightarrow{h} Hom(H_n(C);G) \to 0$$

is *split exact.*

3. The kernel of h.

Let $Z^{\perp n} := \{u \in C^n; u = 0 \text{ on } Z_n\}$. Note $Z^n = \{u \in C^n; u = 0 \text{ on } B_n\}$, so $B^n \subset Z^{\perp n} \subset Z^n$. Since $u_0 = 0$ iff $u = 0$ on Z_n, we have: $ker(h) = Z^{\perp n}/B^n$.

Claim. Let $E = \{f : B_{n-1} \to G \text{ hom}; f \text{ extends to } Z_{n-1}\}$, Then:

$$Z^{\perp n}/B^n \approx Hom(B_{n-1};G)/E.$$

Proof. An isomorphism $\Psi : Z^{\perp n}/B^n \to Hom(B_{n-1};G)/E$ is induced by the hom. $\overline{\Psi} : Z^{\perp n} \to Hom(B_{n-1};G)$, defined as follows:

$$\overline{\Psi}(u) = \bar{u}, \quad \bar{u}(\partial x) = u(x), \quad x \in C_n.$$

This is well-defined, since $\partial x = \partial y \Rightarrow \partial(x-y) = 0 \Rightarrow u(x)-u(y) = u(x-y) = 0$, if u vanishes on Z_n. $\overline{\Psi}$ is clearly surjective: given $\bar{u} \in Hom(B_{n-1};G)$, define $u(x) = \bar{u}(\partial x)$ for $x \in C_n$; clearly u vanishes on Z_n.

We have: $\overline{\Psi}(B^n) = E$:

(i) Assume $u = \delta v$. Then $u = v\partial$, so $\bar{u}(\partial x) = u(x) = v(\partial x)$. Thus $\bar{u} : B_{n-1} \to G$ extends to $v : C_{n-1} \to G$, so $\bar{u} = \Psi(u) \in E$.

(ii) Conversely, assume $\bar{u} : B_{n-1} \to G$ extends to Z_{n-1}. We need to show $u \in B^n$, i.e. $u = \delta v$ for some $v \in C^{n-1}$, i.e. $u(x) = \delta v(x) = v(\partial x)$, for all $x \in C_n$. We have $u(x) = \bar{u}(\partial x)$. Let v extend $\bar{u}$ to C_{n-1}. Then $u(x) = v(\partial x)$, as we wished to show.

Note: $0 \to Z_{n-1} \to C_{n-1} \xrightarrow{\partial} B_{n-2}$ is split exact (since $B_{n-2} \subset C_{n-2}$ is free), so extending to Z_{n-1} is equivalent to extending to C_{n-1}.

Conclusion. Defining $Ext(H_{n-1};G) := Hom(B_{n-1};G)/E$, we have shown (for the singular homology $\mathbb{Z}$ coeficients) and cohomology (G coefficients) of a space X) that the short exact sequence:

$$0 \to Ext(H_{n-1}(X);G) \to H^n(X;G) \xrightarrow{h} Hom(H_n(X);G) \to 0$$

is *split exact*, in particular:

$$H^n(X;G) \approx Hom(H_n(X);G) \oplus Ext(H_{n-1}(X);G),$$

the *universal coefficients theorem* for cohomology.

General definition of $Ext(H;G)$ (H, G abelian groups.)

Consider a *free resolution* of H: an exact sequence $0 \to B \xrightarrow{i} Z \xrightarrow{j} H \to 0$, where Z (and hence B) is a free abelian group. (Free resolutions always exist: H can be described by generators and relations; let Z be the free group defined by the generators, and B the subgroup of Z given by the relations.)

Dualizing does not preserve exactness: surjectivity at the last step fails; we only have that:

$$0 \to Hom(H;G) \xrightarrow{j^T} Hom(Z;G) \xrightarrow{i^T} Hom(B;G)$$

is exact (for a proof, see [Rotman, p.380], including an example where the last hom. i^T is not surjective.) We define:

$$Ext(H;G) := coker(i^T) := Hom(B;G)/E, \text{ where } E := im(i^T) = \{ \text{ hom } B \to G \text{ extending to } Z\}.$$

Then we do have an exact sequence:

$$0 \to Hom(H;G) \xrightarrow{j^T} Hom(Z;G) \xrightarrow{i^T} Hom(B;G) \xrightarrow{\pi} Ext(H;G) \to 0,$$

where π is the quotient projection. We still have to check that the notation makes sense, that is, that $coker(i^T)$ depends only on H (and G), not on the particular free resolution of H chosen.

Consider a second free resolution $0 \to B' \xrightarrow{i'} Z' \xrightarrow{j'} H \to 0$ for H. We may assume $B = ker(j), B' = ker(j')$ are subgroups of Z, Z', and that the homs. i, i' are inclusions. In the following, we adopt the duality notation $Z^* = Hom(Z;G)$ etc. for homomorphism groups (with values in the fixed group G.) Dualizing, we have the exact sequence:

$$0 \to H^* \xrightarrow{j^T} Z^* \xrightarrow{i^T} B^*, \quad E = im(i^T) \subset B^*, \quad Ext = B^*/E.$$

A *morphism* between two free resolutions $(B, i, Z, j, H), (B', i', Z', j', H)$ of H is defined by a homomorphism $\varphi : Z \to Z'$ such that $j'\varphi = j$ and $\varphi(B) \subset B'$. We set $\psi = \varphi_{|B} : B \to B'$. Thus $\varphi i = i'\psi$.

A morphism of free resolutions of H induces a hom. $\varphi^* : Ext' \to Ext$, as follows. We have $\psi^T : B'^* \to B^*$ and $i'\psi = \varphi i$ implies $\psi^T i'^T = i^T \varphi^T$. Thus $\psi^T(E') \subset E$, and passing to the quotients we have a well-defined homomorphism, which we denote by $\varphi^* : Ext' \to Ext$.

It follows from the construction that, given a second morphism $\chi : (B', i', Z', j', H) \to (B'', i'', Z'', j'', H)$, we have $(\chi \circ \varphi)^* = \varphi^* \chi^* : Ext'' \to Ext$.

Lemma. (i) Given two free resolutions $(B, i, Z, j, H), (B', i', Z', j', H)$ of an abelian group H, there exists a morphism $\varphi : Z \to Z'$ between them.

(ii) Any two morphisms φ_0, φ_1 between two given free resolutions of H are related by a homomorphism $D : Z \to B'$ satisfying $i'D = \varphi_0 - \varphi_1$. Letting D_B be the restriction of D to B, we have $D_B = \psi_0 - \psi_1$.

(iii) Any two morphisms φ_0, φ_1 between two given free resolutions of H induce the same homs. $Ext' \to Ext$, that is: $\varphi_0^* = \varphi_1^*$.

(iv) The induced hom. $\varphi^* : Ext' \to Ext$ is always an isomorphism (for any morphism φ.) In particular, $Ext = B^*/E$ depends only on H (and G), up to isomorphism.

Proof. (i) Let $\{e_\alpha\}_{\alpha \in \mathcal{A}}$ be a basis for Z, and $\mathcal{B} \subset \mathcal{A}$ a subset such that $\{e_\alpha\}_{\alpha \in \mathcal{B}}$ is a basis for the subgroup $B \subset Z$. Given e_α, there exists $e'_\alpha \in Z'$ such that $j'e'_\alpha = je_\alpha$ (since j' is surjective.) Set $\phi(e_\alpha) = e'_\alpha$ and extend to all of Z by $\mathbb{Z}$-linearity. Note that if $\alpha \in \mathcal{B}$ we have $j(e_\alpha) = 0$, so $j'(e'_\alpha) = 0$ and $e'_\alpha \in B'$. Thus $\varphi(B) \subset B'$.

(ii) For any $\alpha \in \mathcal{A}$ we have $j'(\varphi_0 - \varphi_1)(e_\alpha) = j(e_\alpha) - j(e_\alpha) = 0$, so $(\varphi_0 - \varphi_1)(e_\alpha) = b'_\alpha$, for some $b'_\alpha \in B'$. Setting $D(e_\alpha) = b'_\alpha$ and extending $\mathbb{Z}$-linearly to all of Z, we have $i'D = \varphi_0 - \varphi_1$. Restricting to the subgroup B, this implies $D_B = \psi_0 - \psi_1$.

(iii) It is enough to show that $im(\psi_0^T - \psi_1^T) \subset E$. We have $\psi_0 - \psi_1 = D_B$ and $D_B = Di$, so $\psi_0^T - \psi_1^T = i^T D^T$ and $im(\psi_0^T - \psi_1^T) \subset im(i^T) = E$.

(iv) From part (i), construct a morphism $\bar{\varphi} : (B', i', Z, j', H) \to (B, i, Z, j, H)$. This gives two morphisms from (B, i, Z, j, H) to itself: the composition $\bar{\varphi} \circ \varphi$ and the identity. From part (iii), $\varphi^* \bar{\varphi}^* = (\bar{\varphi} \circ \varphi)^* = id^* = id_{Ext}$. Analogously, $\bar{\varphi}^* \varphi^* = (\varphi \circ \bar{\varphi})^* = id^* = id_{Ext'}$. Thus φ^* is an isomorphism.

Examples. It follows directly from the definition that, for any abelian G, if H is free abelian we have $Ext(H; G) = 0$. It is also easy to show that:

$$Ext(\oplus H_i; G) = \oplus Ext(H_i; G).$$

Clearly $Hom(\mathbb{Z}; G) \approx G$, and multiplication by n in $\mathbb{Z}$ corresponds to the homomorphism $G \to G$, $g \mapsto ng$.

This gives easy examples of the fact that, in general, $Ext(H; G) \neq Ext(G; H)$.

Problem 1. Show this implies $Ext(\mathbb{Z}_n, G) = G/nG$.
In particular $Ext(\mathbb{Z}_n, \mathbb{Z}) = \mathbb{Z}_n$ and $Ext(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_d$, $d = gcd(n, m)$.

Problem 2. Let $H = \mathbb{Z}^k \oplus T$, where T is a finite Abelian group. Prove that $Ext(H; \mathbb{Z}) \approx T$.

Proposition. If G is an Abelian group such that $nG = G$ for any $n \in \mathbb{N}$, then $Ext(H; G) = 0$ for any Abelian group H.
In particular, this is the case if G is the additive group of $\mathbb{Q}$ or $\mathbb{R}$.

Proof. [Prasolov p.31] Any free resolution (B, i, Z, j, H) of H determines a homomorphism $i^T : Hom(Z; G) \to Hom(B; G)$. We claim i^T is surjective. Let $u \in Hom(B; G)$. Given $y \in Z \setminus B$, extend u to the group generated by B and y as follows: if $ny \notin B$ for all $n \in \mathbb{N}$, set $\tilde{u}(y) = 0$. Otherwise, let n_0 be the least such n; then set $\tilde{u}(y) = g$, where $g \in G$ is such that $n_0 g = u(n_0 y)$.
Now extend to all of Z by induction (assume if needed that Z/B is finitely generated.)

Problem 3. Using the known homology of RP^n (with $\mathbb{Z}$ coefficients), prove that: (i) For n even, $H^p(RP^n) = \mathbb{Z}_2$ if p is even and $2 \leq p \leq n$; o otherwise, except for $p = 0$, where it is $\mathbb{Z}$. (ii) For n odd, the same holds, except for $p = n$, where $H^n(RP^n) = \mathbb{Z}$.

Problem 4. Prove that, for all $n \geq 1$: $H^p(RP^n; \mathbb{Z}_2) = \mathbb{Z}_2$ if $0 \leq p \leq n$.