CLASSIFICATION OF PRINCIPAL BUNDLES AND VECTOR BUNDLES[1]

1. Notation and definitions. $p : E \xrightarrow{Y} X$. E is the total space, X the base, Y the fiber, $E_x = p^{-1}(x)$ the fiber over x. (For $V \subset X$, an alternative notation for $p^{-1}(V)$ is $E_{|V}$.) We have an open cover $\{U_j\}$ of X and local trivializations $\phi_j : U_j \times Y \to p^{-1}(U_j)$, $p(\phi_j(x,y)) = x$. For $x \in U_j$, $y \in Y$, let $\phi_{j,x}(y) = \phi_j(x,y) \in E_x$. We assume a 'structure group' $G \subset Homeo(Y)$, so that the 'transition maps' $g_{ij}(x) = \phi_{j,x}\phi_{i,x}^{-1}$ define continuous maps $g_{ij} : U_i \cap U_j \to G$. We also introduce:

$$p_j : p^{-1}(U_j) \to Y, \quad p_j(e) = \phi_{j,x}^{-1}(e) \in Y \text{ if } e \in E_x, x \in U_j,$$

the point of Y corresponding to $e \in E_x$ in the trivialization ϕ_j.

A *bundle map* from $p : E \xrightarrow{Y} X$ to $p' : E' \xrightarrow{Y} X'$ (same fiber, same structure group G) is a fiber-preserving map $h : E \to E'$, inducing the map $f : X \to X'$. Thus h maps E_x to $E'_{f(x)}$; we write $h_x : E_x \to E'_{f(x)}$, and h_x is assumed to be a *homeomorphism*. Also assumed is the following: suppose $x \in V_j \cap f^{-1}(V'_k)$ and $f(x) = x'$. Then the map:

$$h_{kj}(x) := (\phi'_{k,x'})^{-1} \circ h_x \circ \phi_{j,x} = p'_k \circ h_x \circ \phi_{j,x},$$

a homeomorphism of Y, is assumed to be in G; and the map

$$h_{kj} : V_j \cap f^{-1}(V_k) \to G$$

so defined is continuous ($h_{kj}(x)$ may be thought of as the 'coordinate expression of h_x'.) Sometimes we denote a bundle map as the pair (h, f).

An *isomorphism* between bundles E, E' with the same fiber Y, base space X and structure group G is a bundle map $E \to E'$ inducing the identity on X. Notation: $E \approx E'$.

Pullback bundle. Given $p' : E' \xrightarrow{Y} X'$ and $f : X \to X'$, define a new bundle $p_f : f^*E' \xrightarrow{Y} X$ via:

$$f^*E' = \{(x, e') \in X \times E'; p'(e') = f(x)\}, \quad p_f(x, e') = x.$$

It is easy to see this is again a loc. triv. bundle, with the same fiber and group as E'.

[1] Source: N. Steenrod, *The Topology of Fiber Bundles* (1950, Princeton U.P.), sections 10, 11, 19.

The trivializing cover of f^*E' is the pullback $\{A_j = f^{-1}(V'_j)\}$ of that of E', with trivializations given in terms of those of E',

$$\phi'_j : V'_j \times Y \to (p')^{-1}(V'_j)$$

by:

$$\hat{\phi}_j : A_j \times Y \to p_f^{-1}(A_j) = \{(x, e') \in A_j \times E'; p'(e') = f(x) \in V'_j\},$$

$\hat{\phi}_j(x, y) = (x, \phi'_{j,f(x)}(y)) \in p_f^{-1}(A_j)$ or $\hat{\phi}_{j,x} = \phi'_{j,f(x)} : Y \to (f^*E')_x = E'_{f(x)}$. Also define:

$$\hat{p}_j : p_f^{-1}(A_j) \to Y; \quad \hat{p}_j := (\hat{\phi}_{j,x})^{-1} \text{ on } (f^*E')_x, \quad x \in A_j.$$

There is a close connection between pullback bundles and bundle maps: f^*E' comes endowed with a natural bundle map $(\tilde{f}, f) : f^*E' \to E'$, defined via $\tilde{f}(x, e') = e'$. And conversely, we have:

2. Isomorphism theorem. Let $p : E \xrightarrow{Y} X, p' : E' \xrightarrow{Y} X'$ have the same fiber and structure group. Suppose there exists a bundle map $h : E \to E'$, inducing $f : X \to X'$. Then E is isomorphic to f^*E':

$$(\exists (h, f) : E \to E' \text{ bundle map}) \Rightarrow E \approx f^*E'.$$

In fact there exists a bundle isomorphism $f_0 : E \to f^*E'$ so that h is the composition:

$$E \xrightarrow{f_0} f^*E' \xrightarrow{\tilde{f}} E',$$

so $h = \tilde{f} \circ f_0$, where (f_0, id_X) is an isomorphism $E \approx f^*E'$ and $\tilde{f} : f^*E' \to E'$ the natural map just defined.

Proof. Define:

$$f_0 : E \to f^*E', \quad f_0(e) = (p(e), h(e)).$$

Note we do have $p'h(e) = f(p(e))$, since h induces f on the base; so $f_0(e) \in f^*E'$. Then it is clear from the definition of $\tilde{f}$ that $\tilde{f}f_0(e) = h(e)$, for all $e \in E$. We show f_0 is a bundle map, inducing the identity on the base X (that is, $p_f \circ f_0 = p$, where $p_f(x, e') = x$ for $(x, e') \in f^*E'$). Indeed, for $e \in E$:

$$(p_f \circ f_0)(e) = p_f(p(e), h(e)) = p(e),$$

from the definition of p_f. On the fiber E_x over $x \in X$, f_0 is the map:

$$f_{0x} : E_x \to (f^*E')_x, \quad (x, e) \mapsto (x, h_x(e)),$$

and thus is a homeomorphism, since h, as a bundle map, defines a homeomorphism $h_x : E_x \to E'_{f(x)}$.

It remains to examine the 'coordinate expression' of f_0. We have, with $\hat{p}_k := (\hat{\phi}_{k,x})^{-1} : (f^*E')_x \to Y, \quad x \in f^{-1}(W'_k) \cap V_j$ ($\{V_j\}$ is the trivializing cover for E, $\{W'_k\}$ that for E'):

$$f_{kj}(x)(y) := \hat{p}_k(f_0)_x \phi_{j,x}(y) = \hat{p}_k(x, h\phi_j(x,y))$$

$$= (\hat{\phi}_{k,x})^{-1}(h_x\phi_{j,x})(y) = (\phi'_{k,f(x)})^{-1}(h_x\phi_{j,x})(y) = p'_k h_x \phi_j(x,y) = h_{kj}(y),$$

the same as the coordinate expression for h, and therefore taking values in G. □

3. Homotopy invariance. Let $p : E \xrightarrow{Y} X, p' : E' \xrightarrow{Y} X'$ have the same fiber and group. We recall the *homotopy lifting property* (where X is assumed to be a normal, locally compact, Lindelöf space):

Let $\tilde{h}_0 : E \to E'$ be a bundle map, inducing $h_0 : X \to X'$. Let $H : X \times I \to X'$ be a homotopy of h_0. Then there exists a homotopy $\tilde{H} : E \times I \to E'$, a bundle map lifting H, that is: $p' \circ \tilde{h}_t = h_t \circ p$.

The bundle $E \times I$. Let $p : E \xrightarrow{Y} X$ be a loc.triv. fiber bundle with structure group G. The bundle $q : E \times I \xrightarrow{Y} X \times I$, with group G, has projection map $q(e,t) = (p(e),t)$ and local trivializations $\psi_j : (V_j \times I) \times Y \to q^{-1}(V_j \times I)$ given by:

$$\psi_j(x,t,y) = (\phi_j(x,y),t),$$

where the $\phi_j : V_j \times Y \to p^{-1}(V_j)$ are the local trivializations of E.

Theorem: bundles over $X \times I$. With the same hypotheses on X, any fiber bundle $p' : E' \xrightarrow{Y} X \times I$ is isomorphic to one of the form $E \times I$. Indeed we may take $E = i_0^* E'$, where $i_0 : X \to X \times I$ is the map $i_0(x) = (x,0)$. The total space is:

$$E = i_0^* E' = \{(x,e'); p'(e') = i_0(x) = (x,0)\}$$

We have: $E' \approx (i_0^* E') \times I$.

Proof. Let $E = i_0^* E'$. The identity $i : X \times I \to X \times I, i(x,t) = (x,t)$ is a homotopy of i_0. Lifting it we obtain a bundle map $\tilde{h} : E \times I \to E'$, a homotopy starting from $\tilde{h}(x,e',0) = e'$ where $p'(e') = (x,0)$. Since $\tilde{h}$ induces the identity on $X \times I$, it is a bundle isomorphism. □

Theorem: homotopy invariance of the pullback. With X as before, let $p' : E' \xrightarrow{Y} X'$, and suppose $f_0, f_1 : X \to X'$ are homotopic. Then f_0^*E' and f_1^*E' are isomorphic.

Proof. Let $F : X \times I \to X'$, or $f_t : X \to X'$ be a homotopy from f_0 to f_1. Since F^*E is a bundle over $X \times I$, by the preceding theorem we have an isomorphism:

$$F^*E' \approx E \times I \text{ where } E = i_0^*(F^*E'), \quad i_0 : X \to X \times I, i_0(x) = (x, 0).$$

(E is a bundle over X, with fiber Y.) For each $t \in [0,1]$, let $i_t : X \to X \times I$, $i_t(x) = (x,t)$. Thus $f_t = F \circ i_t$, and we have:

$$f_t^*E' = i_t^*(F^*E') \approx i_t^*(E \times I).$$

On the other hand, from the definition (see above) of the bundle $E \times I$, we know $\tilde{i_t}(e) = (e,t)$ defines, for each $t \in [0,1]$, a bundle map $(\tilde{i_t}, i_t) : E \to E \times I$. By the *isomorphism theorem* above, it follows $(\tilde{i_t})^*(E \times I)$ is isomorphic to E for each $t \in [0,1]$. We conclude:

$$f_0^*E' \approx i_0^*(E \times I) \approx E \approx i_1^*(E \times I) \approx f_1^*E',$$

in particular $f_0^*E' \approx f_1^*E'$. $\square$

In words: the pullback h_t^*E' is isomorphic to the fiber over t of $E \times I$ (a 'constant bundle' in t), and those fibers are all isomorphic to E itself, where E is the pullback of E' under the homotopy F, restricted to $t = 0$.

4. Universal G-principal bundles.

We now specialize to the case of G-principal bundles. First, an important remark:

Remark: We require bundle maps $h : E \to E'$ of principal G-bundles to be equivariant with respect to the right actions of G on the total spaces E, E':

$$h(e \cdot g) = h(e) \cdot g, \forall e \in E.$$

Definition. Let $\pi : \mathcal{E} \xrightarrow{G} \mathcal{B}$ be a principal G-bundle. It is said to be *n-universal* if, given a finite relative cell complex (X, A), of dimension at most n, if we're given a principal G-bundle $p : E \xrightarrow{G} X$ and a partial bundle map $(h, f) : p^{-1}(A) \to \mathcal{E}$, there exists an extension of (h, f) to a bundle map $(h, f) : E \to \mathcal{E}$.

In particular when $A = \emptyset$, this asserts the existence of a map $f : X \to \mathcal{B}$ and a bundle map $(h, f) : E \to \mathcal{E}$ inducing f, implying the isomorphism $E \approx f^*\mathcal{E}$. (For any G-principal bundle E over an n-dimensional celcomplex X.)

Classification theorem. Let $\pi : \mathcal{E} \xrightarrow{G} \mathcal{B}$ be an $(n+1)$-universal G-principal bundle; let X be (homotopy equivalent to) an n-dimensional cell complex.Then the assignment $[f] \mapsto [f^*\mathcal{E}]$ establishes a *bijection*:

$$\Phi : [X, \mathcal{B}] \leftrightarrow Prin_G(X)$$

between homotopy classes of maps $f : X \to \mathcal{B}$ and isomorphism classes of principal G-bundles over X.

Proof. (1) Φ is well-defined: this follows directly from the homotopy invariance theorem;

(2) Φ is surjective: by definition of '$(n+1)$-universal bundle', given $p : E \xrightarrow{G} X$, a G-principal bundle over an n-dimensional cell complex X, there exists a bundle map $(h, f) : E \to \mathcal{E}$. Then the isomorphism theorem implies $E \approx f^*\mathcal{E}$, so $\Phi([f]) = [f^*\mathcal{E}] = [E]$.

(3) Φ is injective: let $f_0, f_1 : X \to \mathcal{B}$ so that $E_0 := f_0^*\mathcal{E} \approx f_1^*\mathcal{E} := E_1$, as G-principal bundles over X. The claim is that $f_0 \simeq f_1$ (homotopic.)

From the definition of pullback bundle, we have natural bundle maps $(h_0, f_0) : E_0 \to \mathcal{E}$, $(h_1, f_1) : E_1 \to \mathcal{E}$. Let $(h, id_X) : E_0 \to E_1$ be the hypothesized isomorphism. Consider the product bundle $E_0^I = E_0 \times I$. It comes with a bundle map $r : E_0^I \to E_0$, namely $r(e, t) = e$, covering $(x, t) \mapsto x$. We have the restrictions of E_0^I to $t = 0, t = 1$, both clearly isomorphic to E_0, as follows:

$$E_0' := E^I_{0|t=0}, \quad E_1' := E^I_{0|t=1}, \quad r_0 : E_0' \approx E_0, \quad r_1 : E_1' \approx E_0,$$

where the isomorphisms (bundle maps covering id_X) are restrictions of r to the endpoints of I: $r_0 = r_{|E_0'}, r_1 = r_{E_1'}$.

We now consider the disjoint union bundle, with a bundle map h' to $\mathcal{E}$, defined by composition using h_0, h and h_1:

$$h' : E_0' \sqcup E_1' \to \mathcal{E}, \quad h' = h_0 r_0 \text{ on } E_0'; \quad h' = h_1 h r_1 \text{ on } E_1'.$$

Note h' induces f_0 on E_0' and f_1 on E_1'.

Now $(X \times 0) \sqcup (X \times 1)$ is a subcomplex of the $(n+1)$-dimensional cell complex $X \times I$. Since $\mathcal{E}$ is assumed $(n+1)$-universal, the bundle map h'

admits an extension to a bundle map $(H,F) : E_0^I \to \mathcal{E}$. The induced map $F : X \times I \to \mathcal{B}$ is a homotopy from f_0 to f_1. $\square$

There is a general existence theorem for universal G-principal bundles, but in many geometric cases they 'exist in nature'. How do we recognize them?

Characterization theorem. $\mathcal{E} \xrightarrow{G} \mathcal{B}$ (G-principal) is n-universal if, and only if, the total space $\mathcal{E}$ is $(n-1)$-connected (i.e. path-connected, with vanishing homotopy groups up to dimension $n-1$: $\pi_k(\mathcal{E}) = \{0\}, k = 1, \ldots, n-1$.)

Proof. (1) Assume $\pi : \mathcal{E} \to \mathcal{B}$ (G-principal) is n-universal. Let $f : S^k \to \mathcal{E}$ be given; the goal is to show that f is homotopically trivial if $k \leq n-1$. We use f and the right G-action on $\mathcal{E}$ to define a bundle map F from the trivial principal bundle $S^k \times G$ to $\mathcal{E}$:

$$F : S^k \times G \to \mathcal{E}, \quad F(y,g) = f(y) \cdot g; \quad F \text{ covers } \pi \circ f : S^k \to \mathcal{B}.$$

Since (D^{k+1}, S^k) is a relative cell complex of dimension $k+1 \leq n$, the n-universal hypothesis implies the bundle map F extends to one from $D^{k+1} \times G$ to $\mathcal{E}$ (we'll still call it F): $F : D^{k+1} \times G \to \mathcal{E}$. Then for $y \in S^k$: $F(y, id_G) = f(y)$, so $F(\cdot, id_G)$ extends f to D^{k+1}, showing $\pi_k(\mathcal{E})$ is trivial.

(2) Prior to showing the converse, we establish the following claim:

(*) assuming $\mathcal{E}$ is $(n-1)$-connected, any bundle map $h : S^k \times G \to \mathcal{E}$ extends to a bundle map $h' : D^{k+1} \times G \to \mathcal{E}$, if $k \leq n-1$.

Indeed, given h, we obtain $f : S^k \to \mathcal{E}$ via $f(y) = h(y, id_G)$, and since $\pi_k(\mathcal{E}) = 0$, this extends to $f' : D^{k+1} \to \mathcal{E}$. Now use the right action of G on $\mathcal{E}$ to define $h'(y,g) = f'(y) \cdot g$, for $y \in D^{k+1}$. This clearly extends h, since for $y \in S^k$:

$$h'(y, id_G) = f'(y) = f(y) = h(y, id_G),$$

and for the other group elements the result follows from G-equivariance of h, h'.

We prove that $\mathcal{E} \xrightarrow{G} \mathcal{B}$ is n-universal, assuming $\mathcal{E}$ is $(n-1)$-connected. Given $E \xrightarrow{G} X$ G-principal, we must find a bundle map $(h,f) : E \to \mathcal{E}$. This is done successively over the skeleta $X^{(i)}$ of X, beginning with $X^{(0)}$ (finitely many vertices.) In the proof we take the subcomplex $A = \emptyset$, for simplicity.

We call a map $\xi : G \to E_x$ *admissible* if $x \in U_j$ and $b_j \circ \xi \in G$ (recall $b_j : E_x \to G$ maps e to the element of G corresponding to e in the trivialization of E over U_j.) So map each $x \in X^{(0)}$ to some (arbitrary) $b \in \mathcal{B}$, then consider two (arbitrary) admissible maps $\xi : G \to E_x$ and $\zeta : G \to \mathcal{E}_b$, and define:

$$h_x = \zeta\xi^{-1} : E_x \to \mathcal{E}_b.$$

(The maps ξ, ζ being admissible guarantees h_x is G-equivariant.). This defines a bundle map $E_{|X^{(0)}} \to \mathcal{E}$.

Inductively, assume h is already defined on the k-skeleton $X^{(k)}$ (where $k < n$), and extend h to each $(k+1)$-cell e, as follows. Since cells are contractible, we have:

$$E_{|\overline{e}} \approx \overline{e} \times G, \quad E_{|\partial e} \approx \partial e \times G.$$

(*Remark:* We need to assume at this point that X is a *regular* cell complex, that is, the characteristic maps $\Phi : \overline{D}^{k+1} \to \overline{e}$ of the cells of X are homeomorphisms up to the closure.)

Since $\partial e \subset X^{(k)}$, h is already defined on $p^{-1}(\partial e)$, with values in $\mathcal{E}$. We use claim (*) established just above, to extend a bundle map to $\mathcal{E}$ from $\partial e \times G$ to $\overline{e} \times G$, hence (by composing with isomorphisms) extending h from $E_{|\partial e}$ to $E_{|\overline{e}}$ (again with values in $\mathcal{E}$). Doing this cell by cell, we extend h to $X^{(k+1)}$, and after finitely many steps to $X^{(n)}$. $\square$

Classification of vector bundles.

1. Vector bundle morphisms and isomorphism.

Let ξ, η be k, l-vector bundles over base spaces $B(\xi), B(\eta)$. A vector bundle morphism is a pair (u, f) of continuous maps, where $u : E(\xi) \to E(\eta)$ preserves fibers and is linear in each fiber, and $f : B(\xi) \to B(\eta)$ is the induced map of base spaces.

If ξ, η have the same base space B, u is a linear isomorphism on fibers (so $k = l$) and the induced map $f = id_B$, we say ξ and η are (strongly) isomorphic (Notation: $\xi \approx \eta$). It is easy to show [M-S Lemma 3.1] this implies $u : E(\xi) \to E(\eta)$ is a homeomorphism.

Let $f : B \to X$, η a k-vector bundle over X. Recall the pullback $f^*\eta$ has total space and projection:

$$E(f^*\eta) = \{(b, v) \in B \times E(\eta); f(b) = p_\eta(v) \in X\}, \quad p(b, v) = b.$$

We have the following easy but useful fact:

Proposition. Suppose $(u, f) : \xi \to \eta$ is a vector bundle morphism and an isomorphism on each fiber. Then $\xi \approx f^*\eta$ (as vector bundles over B.)

For the *proof* we define $h : E(\xi) \to E(f^*\eta)$ via: $h(e) = (p_\xi(e), u(e))$ (note $p_\eta(u(e)) = f(p_\xi(e))$, so this makes sense.) Then h is continuous and maps each fiber $V_b(\xi)$ isomorphically onto $V_b(f^*\eta)$ (since on each fiber $V_b(\xi)$, h coincides with u.)

2. Canonical and universal k-vector bundles.

We denote by γ_k^n the canonical k-plane bundle over the real Grassmannian $G_k(R^n)$ $(n \geq k)$, with total space and projection:

$$E(\gamma_k^n) = \{(X, v) \in G_k(R^n) \times R^n; v \in X.\}, \quad p(X, v) = X.\}$$

The universal k-vector bundle γ_k has as base space the $G_k(R^\infty)$, the infinite increasing union of the $G_k(R^n)$, an inifnite-dimensional CW complex given the 'weak topology', with total space:

$$E(\gamma_k) = \{(X, v) \in G_k(R^\infty) \times R^\infty; v \in X\}, \quad p(X, v) = X.$$

Note $E(\gamma_k)$ is the infinite increasing union of the $E(\gamma_k^n)$, with compatible projection maps.

The canonical bundle γ_k^n over $G_k(R^n)$ is associated with an equally canonical $(n-k)$-vector bundle over the same base, its orthogonal complement $\gamma_k^{n\perp}$, with total space and projection:

$$E(\gamma_k^{m\perp}) = \{(X, v) \in G_k(R^n) \times R^n; v \in X^\perp\}, \quad q(X, v) = X.$$

Note that their Whitney sum is the trivial n-bundle over $G_k(R^n)$:

$$\gamma_k^n \oplus \gamma_k^{n\perp} = \epsilon^n := G_k(R^n) \times R^n.$$

This implies that if ξ is a k-vector bundle over B and $\xi \approx f^*\gamma_k^n$, for some n and some $f : B \to G_k(R^n)$, then ξ admits a 'complement', a $n-k$-vector bundle η over B so that:

$$\xi \oplus \eta \approx \epsilon_B^n := B \times R^n,$$

the trivial n-bundle over B. (Just let $\eta = f^*(\gamma_k^{n\perp})$.)

3. Existence theorem.

Theorem 1. Let ξ be a k-plane bundle over a compact manifold B. Then there exists n and $f : B \to G_k(R^n)$ so that $\xi \approx f^*\gamma_k^n$.

Proof. See [M-S, Lemma 5.3] (extended to paracompact base spaces in Theorem 5.6, with maps to $G_k(R^\infty)$.) In the compact case, we have an open cover of B by finitely many (say N) open sets, over each of which ξ is trivial. The proof gives a map $B \to G_k(R^{kN})$, using partitions of unity in the same way as the proof that (compact) manifolds embed in euclidean spaces of sufficiently large dimension.

4. Homotopy implies isomorphism.

Theorem 2. Suppose $f_0, f_1 : B \to G_k(R^n)$ are homotopic maps (B compact.) Then the pullback bundles are isomorphic: $f_0^* \gamma_k^n \approx f_1^* \gamma_k^n$.

Proof. (Adapted from [Benedetti, p. 100].) Consider the simple linear algebra fact: suppose we have two direct sum decompositions

$$R^n = V' \oplus V = V'' \oplus V.$$

Then we have a canonical isomorphism $\phi : V' \to V'', \phi(v') = v''$ where $v' = v'' + v$ (unique decomposition.)

Let $F : B \times [0,1] \to G_k(R^n)$ be the homotopy from f_0 to f_1, and consider $F^*(\gamma_k^n)$, a k-vector bundle over $B \times [0,1]$. Denote by $V_{p,t}$ its fiber over (p,t), a k-dimensional subspace of R^n depending continuously on (p,t). Observe the following. For any given $t \in [0,1]$, we have:

$$V_{p,t} \cap V_{p,0}^{\perp} = \{0\}, \forall p \in B \Leftrightarrow R^n = V_{p,t} \oplus V_{p,0}^{\perp}, \forall p \in B \Leftrightarrow f_t^*(\gamma_k^n) \approx f_0^*(\gamma_k^n).$$

Indeed it suffices to consider that the linear algebra fact implies the existence of a continuous field of linear isomorphisms:

$$\phi_p : V_{p,t} \to V_{p,0}, \quad p \in B,$$

that is, of a bundle isomorphism $\phi : f_t^*(\gamma_k^n) \approx f_0^*(\gamma_k^n)$. Clearly the set of $t \in [0,1]$ such that $V_{p,t} \cap V_{p,0}^{\perp} = \{0\}, \forall p \in B$ is open in [0,1]. (This condition is equivalent to the orthogonal projection in R^n mapping $V_{p,t}^{\perp}$ isomorphically onto $V_{p,0}^{\perp}$.)

Claim. $\exists \varepsilon > 0$ such that $\forall 0 \le t \le \epsilon, \forall p \in B, V_{p,t} \oplus V_{p,0}^{\perp} = R^n$.

Proof. Otherwise we have a sequence $(p_n, t_n) \to (p_0, 0)$ in $B \times [0,1]$, such that $\dim(V_{p_n,t_n} \cap V_{p_0,0}^{\perp}) > 0$, which in the limit gives $\dim(V_{p_0,0} \cap V_{p_0,0}^{\perp}) > 0$, contradiction.

Thus if we consider the set:

$$\mathcal{G} = \{\varepsilon \in [0,t]; V_{p,t} \cap V_{p,0}^{\perp} = \{0\}, \forall p \in B, \forall 0 \le t \le \varepsilon\} = \{t \in [0,1]; f_t^* \gamma_k^n \approx f_0^* \gamma_k^n, \forall 0 \le t \le \varepsilon\},$$

we see that $\mathcal{G}$ is a an interval $[0, \epsilon_0)$, open on the right unless $\epsilon_0 = \sup \mathcal{G} = 1$.

But in fact $\epsilon_0 \in \mathcal{G}$: let $t_m \in \mathcal{G}, t_m \uparrow \epsilon_0$. Since $R^n = V_{p\epsilon_0} \oplus V_{p\epsilon_0}^{\perp} \forall p$, we have $V_{pt_m} \cap V_{\epsilon_0 p} = \{0\}^{\perp} \forall p$ for m sufficiently large (due to openness of the condition), hence $f_{t_m}^* \gamma_k^n \approx f_{\epsilon_0}^* \gamma_k^n$ for m large, and since $t_m \in \mathcal{G}$ also $f_0^* \gamma_k^n \approx f_{\epsilon_0}^* \gamma_k^n$, so $\epsilon_0 \in \mathcal{G}$. This implies $\epsilon_0 = 1$, or $f_0^* \gamma_k^n \approx f_1^* \gamma_k^n$, as we wished to show.

5. Gauss maps. [Husemoller p.31.] Definition: a Gauss map to R^m for a k-vector bundle ξ is a continuous map $g : E(\xi) \to R^m$ which is a linear monomorphism on each fiber of ξ (so $m \geq k$.)

For example, $q : E(\gamma_k^n) \to R^n, q(X, v) = v$ is a Gauss map. If $(u, f) : \xi^k \to \gamma_k^n$ is a bundle morphism which is isomorphic on fibers, the composition $q \circ u : E(\xi) \to R^n$ is a Gauss map.

Conversely, if ξ is a k-vector bundle $p : E(\xi) \to B$ and $g : E(\xi) \to R^m$ is a Gauss map for ξ, there exists a bundle morphism $(u, f) : \xi \to \gamma_k^m$ such that $q \circ u = g$.

To see this, for $b \in B$ let $f(b) = g(p^{-1}b)$, the image under g of the fiber of ξ over b, a k-dimensional subspace of R^m and hence a point of $G_k(R^m)$; and for $e \in E(\xi)$, let $u(e) = (f(p(e)), g(e)) \in E(\gamma_k^m)$. Using local trivializations, one sees that u and f are continuous and u is isomorphic on fibers. Thus we have the following simple but useful observation, for an arbitrary k-vector bundle $\xi(E, p, B)$:

$$[\exists f : B \to G_k(R^m), \xi \approx f^*(\gamma_k^m)] \Leftrightarrow [\exists g : E \to R^m \text{ Gauss map}].$$

In particular, it follows from Theorem 1 that any vector bundle over a compact (or paracompact) base admits a Gauss map.

6. The even-odd trick. [Husemoller p. 33, M-S p. 67, Thm 5.7]

To finish the classification theorem, it remains to prove that isomorphism implies homotopy: if $f_0, f_1 : B \to G_k(R^n)$ yield isomorphic vector bundles under pullback: $f_0^* \gamma_k^n \approx f_1^* \gamma_k^n$, then they are homotopic: $f_0 \simeq f_1$ as maps to $G_k(R^n)$ (that is, with the homotopy taking values in $G_k(R^n)$). Unfortunately this is not what is proved, and here is the problem: say $f_0^* \gamma_k^n \approx f_1^* \gamma_k^n \approx \xi$ a k-vector bundle over B, the isomorphisms being given by $u_0, u_1 : E(\xi) \to E(\gamma_k^n)$, inducing f_0, f_1. It would be enough to produce a homotopy between the corresponding Gauss maps $g_0 = q \circ u_0, g_1 = q \circ u_1$, an easier problem since we may try the linear homotopy in R^n:

$$g_t(e) = (1-t) g_0(e) + t g_1(e) \in R^n, \quad e \in E(\xi), t \in [0, 1].$$

Unfortunately there is no way to guarantee this always gives a nonzero vector if $e \neq 0$; that is, that each g_t is itself a Gauss map.

Thus a tricky detour is necessary, which in the end results in a homotopy from f_0 to f_1, but taking values in $G_k(R^{2n})$, not $G_k(R^n)$.

Consider the 'even and odd subspaces' of R^∞:

$$R^{ev} = \{x \in R^\infty; x_{2i+1} = 0 \forall i \geq 0\}; \quad R^{od} = \{x \in R^\infty; x_{2i} = 0 \forall i \geq 0\}.$$

For each $t \in [0,1]$, consider the linear embeddings:

$$k_t^e : R^n \to R^{2n}, k_t^o : R^n \to R^{2n} :$$

$$k_t^e(x_0, x_1, \ldots, x_{n-1}) = (1-t)(x_0, x_1, \ldots, x_{n-1}, 0, \ldots, 0) + t(x_0, 0, x_1, 0, \ldots, x_{n-1}, 0).$$

$$k_t^o(x_0, x_1, \ldots, x_{n-1}) = (1-t)(x_0, x_1, \ldots, x_{n-1}, 0, \ldots, 0) + t(0, x_0, 0, x_1, \ldots, 0, x_{n-1}).$$

We see that each k_t^e, k_t^o ($t \in [0,1]$) is a linear embedding and:

(1) $k_0^e = k_0^o$ is the standard inclusion $R^n \to R^{2n}$ (set the last n coordinates equal to 0).

(2) $k_1^e(R^n) = R^{2n} \cap R^{ev}$, $k_1^0(R^n) = R^{2n} \cap R^{odd}$.

(3) Denote by $q_n : E(\gamma_k^n) \to R^n, q_{2n} : E(\gamma_k^{2n}) \to R^{2n}$ the canonical Gauss maps. Then $k_1^e \circ q_n, k_1^o \circ q_n$ are Gauss maps for γ_k^n, taking values in R^{2n}. Thus, as seen above, there exist vector bundle morphisms (injective on fibers):

$$(u^e, f^e) : \gamma_k^n \to \gamma_k^{2n}, \quad (u^o, f^o) : \gamma_k^n \to \gamma_k^{2n},$$

such that:

$$k_1^e \circ q_n = q_{2n} \circ u^e, \quad k_1^o \circ q_n = q_{2n} \circ u^o.$$

(4) $f^e, f^o : G_k(R^n) \to G_k(R^{2n})$ are homotopic to the standard inclusion maps $j : G_k(R^n) \to G_k(R^{2n})$. Namely,

$$X \mapsto k_t^e(X) \in G_k(R^{2n}), X \in G_k(R^n) \text{ joins } j \text{ at } t = 0 \text{ to} f^e \text{ at } t = 1;$$

$$X \mapsto k_t^o(X) \in G_k(R^{2n}), X \in G_k(R^n) \text{ joins } j \text{ at } t = 0 \text{ to} f^o \text{ at } t = 1.$$

7. From isomorphism to homotopy.

Theorem 3. Let $f_0, f_1 : B \to G_k(R^n)$ such that $f_0^*(\gamma_k^n) \approx f_1^*(\gamma_k^n)$. Then $j \circ f_0 \simeq j \circ f_1$ (homotopic), where $j : G_k(R^n) \to G_k(R^{2n})$ is the standard inclusion.

Proof. (cp. [Husemoller, p. 35].) By hypothesis there exists a k-vector bundle ξ over B and bundle morphisms (isomorphic on fibers):

$$(u_0, f_0) : \xi \to \gamma_k^n, \quad (u_1, f_1) : \xi \to \gamma_k^n$$

and Gauss maps:

$$h_0 = q_n \circ u_0 : E(\xi) \to R^n, \quad h_1 = q_n \circ u_1 : E(\xi) \to R^n.$$

Composing with the maps obtained in the previous subsection, we have:

$$(u^e \circ u_0, f^e \circ f_0) : \xi \to \gamma_k^{2n} \text{ with Gauss map } k_1^e \circ h_0 : E(\xi) \to R^{2n} \cap R^{ev},$$

$$(u^o \circ u_1, f^o \circ f_1) : \xi \to \gamma_k^{2n} \text{ with Gauss map } k_1^o \circ h_1 : E(\xi) \to R^{2n} \cap R^{odd}.$$

Define the map $h_t : E(\xi) \to R^{2n}$, $t \in [0, 1]$:

$$h_t(e) = (1 - t)(k_1^e \circ h_0)(e) + t(k_1^o \circ h_1)(e), \quad e \in E(\xi).$$

Then h_t is a Gauss map for ξ, for each t: in each fiber $V_b(\xi)$, $k_1^e \circ h_0 : V_b(\xi) \to R^{ev}, k_1^e \circ h_1 : V_b(\xi) \to R^{odd}$ are both injective, taking values in subspaces intersecting only at 0; thus h_t is also injective.

This implies there exists a continuous one-parameter family of bundle morphisms:

$$(w_t, \phi_t) : \xi \to \gamma_k^{2n},$$

where $\phi_t : B \to G_k(R^{2n})$ is a homotopy $f^e \circ f_0 \simeq f^o \circ f_1$.

Now recall (from point (4) above): $j \circ f_0 \simeq f^e \circ f_0, j \circ f_1 \simeq f^o \circ f_1$. Thus $j \circ f_0 \simeq j \circ f_1$, as we wished to show.

8. Summary. Theorems 1,2,3 may be summarized as the classification theorem:

Any real k-vector bundle ξ over a compact (or paracompact) base space B is the pullback $f^*(\gamma_k^n)$ under a map $f : B \to G_k(R^n)$, for some n. Homotopic maps $B \to G_k(R^n)$ induce isomorphic bundles; conversely, if $f^*\gamma_k^n \approx g^*\gamma_k^n$, then f is homotopic to g (as maps to $G_k(R^{2n})$). In symbols, the pullback of γ_k over $G_k(R^\infty)$ establishes a bijection:

$$Vect_k(B) \leftrightarrow [B, G_k(R^\infty)].$$

(Isomorphism classes of k-vector bundles over B on the left, homotopy classes of maps on the right.)

The same is true for complex vector bundles:

$$Vect_k^{\mathbb{C}}(B) \leftrightarrow [B, G_k(\mathbb{C}^\infty)].$$

In particular for line bundles:

$$Vect_1(B) \leftrightarrow [B, RP(\infty))] = [B, K(\mathbb{Z}_2, 1)] \leftrightarrow H^1(B; \mathbb{Z}_2).$$

$$Vect_1^{\mathbb{C}}(B) \leftrightarrow [B, \mathbb{C}P(\infty)] = [B, K(\mathbb{Z}, 2)] \leftrightarrow H^2(B; \mathbb{Z}).$$

The last correspondences are given by the first Stiefel-Whitney class (resp. the first Chern class).