H. HOPF'S THEOREM ON CLASSIFICATION OF MAPS $M^n \to S^n$ UP TO HOMOTOPY[1].

We consider two compact oriented n-manifolds without boundary M, N (N connected) and smooth maps $f : M \to N$. Recall that, for a regular value $y \in N$ of f, with preimage $f^{-1}(y) = \{x_1, \dots, x_N\}$, the degree of f satisfies:

$$deg(f) = \sum_{x \in f^{-1}(y)} deg_x f,$$

where f is a local diffeomorphism from a neighborhood of f to one of y, and $deg_x f = \pm 1$, according to whether f preserves (+1) or reverses (-1) orientation at x.

We first recall the proof that if $M = \partial W$ for a compact, oriented $(n+1)$-manifold with boundary W, and f extends to a smooth map $W \to N$, then $deg(f) = 0$. This follows from a geometric observation about orientations.

Lemma 1. Let (W, ω) be a compact, oriented manifold with boundary $M = \partial W$, where ω is the orientation of W and M has the boundary orientation $\partial\omega$ defined by the outward normal. Suppose $K \subset W$ is an embedded one-manifold with boundary (a smooth embedded arc in W), intersecting M transversely at its endpoints $\{P, Q\} = \partial K$. Denote by κ the orientation of K from P to Q and by (ν, ω_ν) the normal bundle of K in W, with the orientation ω_ν defined by κ and ω. We have:

$$\omega_\nu(P) = \partial\omega(P) \Leftrightarrow \omega_\nu(Q) = -\partial\omega(Q).$$

Remark: We assume the Riemannian metric used to define $\nu = \bigcup_{x \in K} \nu_x$ is a local product near P, Q, so that $\nu_P = T_P M, \nu_Q = T_Q M$.

Proof. Let X_P, X_Q be tangent vectors to K at P, Q, belonging to κ_P, κ_Q. Then X_P is inward iff X_Q is outward, which is equivalent to the lemma.

Proposition 1. With the same notation as Lemma 1 ($M = \partial W$ with the boundary orientation $\partial\omega$ and $dim(W) = n+1$), let (N, θ) be a compact, connected, oriented n-manifold, and let $h : W \to N$ be a smooth map. Then $deg(h_{|M}) = 0$.

Proof. Let $y \in N$ be a regular value, simultaneously for h and for $h_{|M}$. Then $h^{-1}(y)$ is a compact one-dimensional embedded submanifold of W, with boundary equal to its intersection with M, and intersecting M transversely. Let K be a connected component of $f^{-1}(y)$ intersecting

[1]Following the proof given in [Hirsch], *Differential Topology*, section 5.1.

$\partial W = M$. K is an embedded arc in W intersecting M transversely at its endpoints P, Q.

Let $\nu = \bigcup_{x \in K} \nu_x$ be the normal bundle of K in W, with respect to a Riemannian metric chosen so that $\nu_P = T_P M, \nu_Q = T_Q M$. For $x \in K$, $df(x)$ induces a linear isomorphism $\Phi_x : \nu_x \to T_y N$.

Denote by κ the orientation of K from P to Q. Endow ν with the orientation ω_ν induced by by ω and κ, chosen so that $\omega_\nu(Q) = \partial\omega(Q)$ (and therefore $\omega_\nu(P) = -\partial\omega(P)$ by Lemma 1, where $\partial\omega$ is the orientation of M induced by the orientation ω of W and the outward normal.) Suppose $Q \in f^{-1}(y)$ is of positive type for $f = h_{|M}$. So:

$$\Phi_Q[\omega_\nu(Q)] = df(Q)[\omega_\nu(Q)] = df(Q)[\partial\omega](Q) = \theta_y,$$

thus by continuity $\Phi_x[\omega_\nu(x)] = \theta_y$ for all $x \in K$, in particular $\Phi_P[\omega_\nu(P)] = \theta_y$. This implies:

$$df(P)[\partial\omega](P) = \Phi_P[-\omega_\nu(P)] = -\Phi_P[\omega_\nu(P)] = -\theta_y,$$

so P is of negative type for f.

Thus at each joint regular value y of h and f, we see that f has equal numbers of preimages of positive and negative type, and hence $deg(f) = 0$.

Review of tubular neighborhoods. Let W be a manifold of dimension $n+1$ (without boundary), $L \subset W$ a compact embedded submanifold, of dimension $0 \leq l \leq n$. Assume W is endowed with a riemannian metric. If L is compact, we may find $\epsilon > 0$ so that the normal ϵ-disk bundle of L defines an open neighborhood $\mathcal{N}$ of L in W, a *normal tubular neighborhood* of L in W:

$$\mathcal{N} = \sqcup_{x \in L} D^{\perp}_{\epsilon}(x).$$

(If you change the metric, the neighborhood changes slightly; hence the indefinite article.) The $(n+1-l)$-dimensional open disks $D^{\perp}_{\epsilon}(x)$ are all disjoint, and nearest-point projection along the normal disks defines a smooth retraction

$$r : \mathcal{N} \to L.$$

If L is noncompact, but properly embedded in W, this is still true, but we have to allow the radius to depend on x: $\mathcal{N} = \sqcup_{x \in L} D^{\perp}_{\epsilon(x)}(x)$.

If W is a manifold with boundary $\partial W = M$ and L is also a manifold with boundary $\partial L = L \cap \partial W$, and transversal to M along ∂L, then we may add the requirement that $\mathcal{N} \cap M$ is a normal tubular neighborhood of ∂L in M (provided the metric is a product near the boundary).

Conversely, we have the following *extension theorem*: for manifolds with boundary $\partial W = M$, if L is a submanifold with boundary of W (as above) and $\mathcal{T}$ is a normal tubular neighborhood of ∂L in M, then we may find a normal tubular neighborhood $\mathcal{N}$ of L in W so that $\mathcal{N} \cap M = \mathcal{T}$. (For proofs of these results see Hirsch, *Differential Topology*, Ch. 4, sect. 5.)

Our main goal is to prove that if $f : M \to S^n$ (smooth) has degree zero (where $M = \partial W$ is n-dimensional and W is compact oriented), then f extends to a smooth map $W \to S^n$. The main step is the following lemma. We follow [Hirsch] in calling a one-dimensional, connected embedded submanifold of W meeting ∂W transversely a *neat arc*.

Extension Lemma: Let W^{n+1} be compact oriented, with boundary $\partial W = M$. Let $K \subset W$ be a neat arc, with endpoints $P, Q \in M$. Let $V = V_0 \sqcup V_1 \subset M$ be an open neighborhood of $\{P, Q\}$ in M (V_0 nbd. of P, V_1 nbd of Q).

Suppose $f : V \to N$ is a smooth map (where N^n is compact, oriented, connected, without boundary) and Let $y \in N$ be a regular value of f, such that $f^{-1}(y) = \{P, Q\}$. Assume f has local degrees with *opposite signs* at P, Q.

Then we may find $W_0 \subset W$, an open tubular neighborhood of K in W, and a smooth map $g : W_0 \to N$ so that : (a)$g = f$ on $W_0 \cap V$; (b) y is a regular value of g; (c) $g^{-1}(y) = K$.

The following standard differential topology result is used in the proof:

Lemma 2. Let $f : U' \to U'$ be a diffeomorphism of an open neighborhood U' of $0 \in \mathbb{R}^n$, $f(0) = 0$. Let $L = df(0) \in GL_n$. Then there exists a diffeomorphism φ of a smaller neighborhood $U \subset U'$ of 0 so that $\varphi(0) = 0, d\varphi(0) = \mathbb{I}$ and $f \circ \varphi = L$ on U.

Proof of extension lemma. We may choose tubular neighborhoods $U_0 \subset V_0, U_1 \subset V_1, N' \subset N$ of P, Q, y (resp.) so that f restricts to diffeomorphisms:

$$f_0 : (U_0, P) \xrightarrow{\sim} (N', y), \quad f_1 : (U_1, Q) \xrightarrow{\sim} (N', y),$$

and further pick local charts at P, Q, y (diffeomorphisms):

$$\phi_0 : (U_0, P) \xrightarrow{\sim} (R^n, 0), \quad \phi_1 : (U_1, Q) \xrightarrow{\sim} (R^n, 0), \quad \psi : (N', y) \xrightarrow{\sim} (R^n, 0).$$

In addition, composing on the right with a further diffeomorphism (as in

Lemma 2) we may assume the compositions:

$$F_0 = \psi \circ f_0 \circ \phi_0^{-1}, \quad F_1 = \psi \circ f_1 \circ \phi_1^{-1} : (R^n, 0) \xrightarrow{\sim} (R^n, 0)$$

are invertible linear maps: $F_0, F_1 \in GL_n$. Consider now the effect on orientations: let Θ_n demote the standard orientation of R^n. Denoting by $\partial\omega$ the boundary orientation induced on $M = \partial W$ by the orientation ω in W, and by θ the orientation of N, we may require ϕ_0, ϕ_1, ψ to be orientation-preserving:

$$\phi_0[\partial\omega] = \phi_1[\partial\omega] = \Theta_n, \quad \psi[\theta] = \Theta_n.$$

Using the extension theorem for tubular neighborhoods, we find a tubular neighborhood $W_0 \subset W$ of K in W, restricting to U_0, U_1 at P, Q (resp.) Further, since K is one-dimensional, the topology of the situation is standard: we may find a diffeomorphism:

$$\phi : (W_0, K) \xrightarrow{\sim} (I \times R^n, 0 \times R^n).$$

We might be inclined to assert $\phi_{|U_0} = \phi_0, \quad \phi_{|U_1} = \phi_1$ (identifying $R^n \times 0, R^n \times 1$ with R^n); but consideration of orientations reveals this isn't quite right. Let κ be the orientation of K from P to Q; together with ω this induces the orientation ω_ν on the normal disk bundle $\nu = \bigcup_{t \in I} \nu_t$ of K, and we want $d\phi$ to satisfy:

$$d\phi : \bigcup_{t \in I} \nu_t \to I \times R^n, \quad \kappa \otimes \omega_\nu \mapsto \partial_t \otimes \Theta_n, \quad \omega = \kappa \otimes \omega_\nu.$$

(denoting by ∂_t the orientation of $I = [0,1]$ from 0 to 1.) Now suppose we require the induced normal orientation at Q to be $\partial\omega(Q)$. Then by lemma 1 we must have:

$$\omega_\nu(Q) = \partial\omega(Q), \quad \omega_\nu(P) = -\partial\omega(P).$$

Thus the orientation $\partial\omega$ on U_0, U_1 coincides at Q with the restriction of ω_ν to $T_Q M$, but at P it is *the opposite* of the restriction of ω_ν to $T_P M$. So the restriction of the diffeomorphism ϕ to U_0 is not the chart ϕ_0 (which we assumed to be orientation-preserving, for the orientation $\partial\omega$ on U_0).

To remedy this we consider a reflection R in R^n and let $\overline{\phi_0} = R\phi_0$, and then we have:

$$\phi_{|U_0} = \overline{\phi_0}, \quad \phi_{|U_1} = \phi_1.$$

Recall now the hypothesis that P, Q are of opposite signs for df, say:

$$df(Q) : \partial\omega(Q) \mapsto \theta_y, \quad df(P) : \partial\omega(P) \mapsto -\theta_y,$$

which imply: $F_1 \in GL_n^+, F_0 \in GL_n^-$. Since ϕ restricts to $\overline{\phi_0}$ at P, instead of F_0 we consider:

$$\bar{F}_0 = \psi \circ f_0 \circ (\overline{\phi_0})^{-1} = F_0 R,$$

so $\bar{F}_0 \in GL_n^+$. Thus $\bar{F}_0$ and F_1 can be connected in GL_n^+ by a smooth curve $F_t, t \in [0,1]$. We may extend the map defined by $\bar{F}_0, F_1$ on $(R^n \times 0) \sqcup (R^n \times 1)$ to $I \times R^n$ via:

$$G : I \times R^n \to R^n, \quad G(t,x) = F_t x;$$

and then the desired extension $g : W_0 \to N$ of f is given by: $g = \psi^{-1} \circ G \circ \phi$.

Condition (b) in the conclusion of the lemma follows from the fact $0 \in R^n$ is a regular value of G (since $F_t \in GL_n$). Condition (c) follows from $G^{-1}(0) = \{(t,0); t \in I\}$. As for condition (a), we have, if $x \in U_1$:

$$g(x) = \psi^{-1} \circ G \circ \phi_1(x) = \psi^{-1} \circ F_1 \circ \phi_1(x) = \psi^{-1}\psi \circ f \circ \phi_1^{-1}\phi_1(x) = f(x),$$

while if $x \in U_0$:

$$g(x) = \psi^{-1} \circ G \circ \overline{\phi_0}(x) = \psi^{-1}\bar{F}_0\overline{\phi_0}(x) = \psi^{-1}F_0 R R \phi_0(x) = \psi^{-1}F_0\phi_0(x) = f_0(x).$$

This concludes the proof of the extension lemma. Note the oriented manifold N is arbitrary at this point.

Degree zero extension theorem. Let (W,ω) be a compact oriented (n+1)-dimensional manifold with boundary $\partial W = M$, with the boundary orientation $\partial\omega$ (outward normal.) Let $f : M \to S^n$ be a smooth map. Then if $deg(f) = 0$, f extends to a continuous map $\bar{f} : W \to S^n$.

Proof. Let $y \in S^n$ be a regular value of f. By the degree hypothesis, the finite set $f^{-1}(y)$ has equal numbers of points of (+) and (-) type. Thus we may find finitely many disjoint embedded oriented neat arcs $K_1, \ldots, K_m$ in W, each K_i connecting a (-) point in $f^{-1}(y)$ to a (+) point. (See [Hirsch], p.126 for the geometric argument)

By the *extension lemma* just proved, there exists $W_0 \subset W$ open neighborhood of $K = \sqcup_i K_i$ and $g : W_0 \to S^n$ agreeing with f on $\partial W_0 \cap M$, with y as a regular value, and such that $g^{-1}(y) = K$. Let $U \subset W$ be a smaller open neighborhood of K, such that $\overline{U} \subset W_0$ and $\partial U \subset W_0 \setminus K$.

Let $X = \partial U \cup (M \setminus U)$; note X is a closed subset of $W \setminus U$. Define $h : X \to S^n \setminus \{y\}$ via:

$$h = g \text{ on } \partial U; \quad h = f \text{ on } M \setminus U.$$

By the Tietze extension theorem, since X is closed in $W \setminus U$, h extends continuously to $H : W \setminus U \to S^n \setminus \{y\}$. Now the desired extension is given by:

$$\bar{f} : W \to S^n; \quad \bar{f} = H \text{ on } W \setminus U; \quad \bar{f} = g \text{ on } U.$$

Since $H = g$ on ∂U, $\bar{f}$ is continuous on W. And $\bar{f} = f$ on $\partial W = M$, since $\bar{f} = H = h = f$ on $M \setminus U$ and $\bar{f} = g = f$ on $U \cap M$.

Applying the theorem to $W = M \times [0, 1]$, we have the homotopy classification theorem:

Corollary. Let $f, g : M^n \to S^n$, where M^n is compact, oriented, without boundary. Then:

$$deg(f) = deg(g) \Rightarrow f \simeq g.$$