MATH 664, SPRING 2026–PROBLEM SET 1

1. (i) Show that a CW complex is contractible if it is an increasing union of subcomplexes $X_1 \subset X_2 \subset \dots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic.

(ii) Show that $S^\infty = \bigcup_{n\geq 0} S^n$ is contractible. (Increasing union via equatorial inclusion $S^n \hookrightarrow S^{n+1}$.)

2. Show that if $f : X \to Y$ is a homotopy equivalence, then the induced homomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ are isomorphisms for all n.

3. It $p : \tilde{X} \to X$ is a covering map, $A \subset X, \tilde{A} = p^{-1}(A), x_0 \in A$ and $p(\tilde{x_0}) = x_0$, then $p_* : \pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \to \pi_n(X, A, x_0)$ is an isomorphism for all $n \geq 2$.

4. Show that an n-connected, n-dimensional CW complex is contractible.

5. Show that a CW complex deformation retracts onto any contractible subcomplex. *Hint:* Homotopy extension property.

6. Show that if X and Y are CW complexes, with X m-connected and Y n-connected, then the pair $(X \times Y, X \vee Y)$ is $(m + n + 1)$-connected.

7. Show that the set of free homotopy classes $[X, Y]$ is finite if X is a finite connected CW complex and $\pi_p(Y)$ is finite for all $p \leq dim(X)$.

8. Show that a map $f : X \to Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ to the universal covers induces isomorphisms on homology groups.

9. Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces isomorphisms on π_p for all $p \leq n$. *Hint:* Pass to the universal cover and use homology.

10. If an n-dimensional CW complex X contains a subcomplex Y homotopy equivalent to S^n, show that the map $\pi_n(Y) \to \pi_n(X)$ induced by inclusion is injective. *Hint:* Hurewicz isomorphism.

11. Show that the spaces $S^n \times \mathbb{R}P^m$ and $S^m \times \mathbb{R}P^n$ $(m \neq n)$ have the same homotopy groups, but are not homotopy equivalent.

12. Prove that the space of all unordered sets of n points in $\mathbb{R}^\infty$ (or S^∞) is a $K(S_n, 1)$, where S_n is the symmetric group.

13. Show that the spaces $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but nonisomorphic homotopy groups.