1. Linearization of Ricci and scalar curvature.

We consider variations of a background metric b. All covariant derivatives, inner products and traces are with respect to b (Levi-Civita connection), unless noted otherwise. *Covention: the Laplacian of a function is the trace of its Hessian.*

Step1. Let $g_t = b + th$, $h \in Sym^2_M$. The difference of L-C connections is a (2,1) symmetric tensor:

$$\Gamma^t(X,Y) = D^t_X Y - D^b_X Y,$$

and we denote by $\dot{\Gamma}$ its first variation. To compute the variation in curvature, define also the (3,1) tensor:

$$\Gamma^{2t}(X,Y,Z) = \Gamma(X,\Gamma(Y,Z)),$$

and then we have:

$$R^t(X,Y)Z - R^b(X,Y)Z = (D_X\Gamma^t)(Y,Z) - (D_Y\Gamma^t)(X,Z) - (\Gamma^{2t}(X,Y,Z) - \Gamma^{2t}(Y,X,Z)).$$

Since $\Gamma = 0$ at $t = 0$, we have for the variation of the (3,1) Riemann tensor:

$$\dot{R}(X,Y)Z = (D_X\dot{\Gamma})(Y,Z) - (D_Y\dot{\Gamma})(X,Z).$$

The Koszul formula gives the first variation of Γ:

$$\langle\dot{\Gamma}(X,Y),Z\rangle = \frac{1}{2}[(D_Xh)(Y,Z) + (D_Yh)(X,Z) - (D_Zh)(X,Y)].$$

Step 2: variation of Ricci. From now on, assume all vector fields in sight have zero covariant derivative at a fixed point $p \in M$. For the (4,0) Riemann tensor, we find:

$$\langle\dot{R}(X,Y)Z,W\rangle = X(\langle\dot{\Gamma}(Y,Z),W\rangle) - Y(\langle\dot{\Gamma}(X,Z),W\rangle)$$

$$= \frac{1}{2}X[(D_Yh)(Z,W) + (D_Zh)(Y,W) - (D_Wh)(Y,Z)] - (Y \leftrightarrow X)$$

$$= \frac{1}{2}[(D_XD_Yh)(Z,W) + (D_XD_Zh)(Y,W) - (D_XD_Wh)(Y,Z)] - (Y \leftrightarrow X).$$

Now take trace over Y, Z: let (e_i) be a local orthonormal frame, normal at p (sum over repeated indices implicit in what follows):

$$\langle\dot{R}(X,e_i)e_i,W\rangle = \frac{1}{2}[(D_XD_{e_i}h)(e_i,W) + (D_xD_{e_i}h)(e_i,W) - (D_XD_Wh)(e_i,e_i)]$$

$$-\frac{1}{2}[(D_{e_i}D_Xh)(e_i,W) + (D_{e_i}D_{e_i}h)(X,W) - (D_{e_i}D_Wh)(X,e_i)].$$

Now replace the last term in the second line by: $-(D_W D_{e_i} h)(X, e_i) - (R(W, e_i)h)(X, e_i)$. Rearranging terms, we find:

$$\dot{Ric}(X, W) = -\frac{1}{2}(D_{e_i} D_{e_i} h)(X, W) - \frac{1}{2}(D_X D_W h)(e_i, e_i)$$
$$+\frac{1}{2}(D_X D_{e_i} h)(e_i, W) + \frac{1}{2}(D_W D_{e_i} h)(e_i, X)$$
$$+\frac{1}{2}[R(X, e_i)h](e_i, W) + \frac{1}{2}[R(W, e_i)h](e_i, X).$$

Step 3. We now express the various terms as geometric differential operators. First we have:

$$\delta : Sym^2_M \to \Omega^1(M), \quad (\delta h)(W) = -(D_{e_i} h)(e_i, W).$$

and its formal adjoint $\delta^* : \Omega^1_M \to Sym^2_M$, the symmetrized covariant derivative:

$$(\delta^* \omega)(X, Y) = \frac{1}{2}[(D_X \omega)(Y) + (D_Y \omega)(X)], \quad \omega \in \Omega^1_M.$$

We easily find then:

$$\delta^*(\delta h)(X, W) = -\frac{1}{2}[(D_X D_{e_i} h)(e_i, W) + (D_W D_{e_i} h)(e_i, X)].$$

To understand the curvature terms, recall how $R(X, Y)$ acts on symmetric bilinear forms:

$$[R(X, e_i)h](e_i, W) = h(R(X, e_i)e_i, W) + h(e_i, R(X, e_i)W)$$
$$= Ric(X, e_j)h(e_j, W) - h(R(e_i, X)W, e_i);$$

and similarly for the term obtained by symmetrizing this in (X, W).

Recall the symmetric product of two symmetric bilinear forms defined by:

$$(k \circ h)(X, W) = k(X, e_i)h(e_i, W), \quad k, h \in Sym^2_M$$

And also the 'action of the Riemann tensor on symmetric bilinear forms':

$$\mathcal{R}[h](X, W) = h(R(e_i, X)W, e_i) = h(R(e_i, W)X, e_i)$$

(cp. [Besse, 1.131(b)]; it's like taking a Ricci trace, but using h instead of the metric.) With these definitions, the curvature terms become:

$$\frac{1}{2}[R(X, e_i)h](e_i, W) + \frac{1}{2}[R(W, e_i)h](e_i, X) = \frac{1}{2}(Ric \circ h + h \circ Ric)(X, W) - \mathcal{R}[h](X, W).$$

Putting everything together, we find for the variation of Ricci:

$$\dot{Ric}[h] = -\frac{1}{2}D^2_{e_i, e_i} h - \frac{1}{2}Hess(tr_b h) - \delta^*(\delta h) + \frac{1}{2}[Ric \circ h + h \circ Ric] - \mathcal{R}[h].$$

(cp. [Besse, 1.180a]).

Variation of Scalar curvature. From $Scal^{g_t} = tr_{g_t} Ric^{g_t}$ follows:

$$\dot{Scal}[h] = -\langle h, Ric_b\rangle_b + tr_b(\dot{Ric}[h]).$$

And then it turns out (easily checked) that the curvature terms in the variation of Ric, combined, have zero trace! We also have:

$$\delta^*(\delta h)(e_j, e_j) = [D_{e_j}(\delta h)](e_j) = -\delta(\delta h).$$

The traces of $D^2_{e_i,e_i}h$ and of $Hess(tr_b h)$ are both equal to $\Delta(tr_b h)$. We conclude:

$$\dot{Scal}[h] = -\Delta(tr_b h) + \delta(\delta h) - \langle Ric, h\rangle_b.$$

(cp. [Besse, 1.174e], where their convention for the Laplacian on functions has the opposite sign to ours.)

2. Adjoints. We have the linearization of scalar curvature at a background metric b:

$$L_b : Sym^2_M \to C^\infty_M, \quad L_b[h] = -\Delta(tr_b h) + \delta(\delta h) - \langle Ric, h\rangle_b$$

and wish to compute its formal adjoint $L_b^* : C^\infty_M \to Sym^2_M$. For two of the terms, this is clear. On the other hand, if either V or h have compact support in M:

$$\int_M V\delta(\delta h)d\mu_b = \int_M \langle dV, \delta h\rangle d\mu_b = \int_M \langle \delta^*(dV), h\rangle d\mu_b = \int_M \langle Hess(V), h\rangle d\mu_b.$$

Here we used the fact δ (on one-forms) is the formal adjoint of the exterior differential d, δ^* (the symmetrized covariant derivative, taking Ω^1_M to Sym^2_M) the formal adjoint of $\delta : Sym^2_M \to \Omega^1_M$, and:

$$(\delta^* dV)(X, Y) = \frac{1}{2}[(D_X dV)Y + (D_Y dV)X] = Hess(V)(X, Y).$$

We conclude the formal adjoint is:

$$L_b^* : C^\infty_M \to Sym^2_M, \quad L_b^*[V] = Hess_b(V) - (\Delta V)b - V Ric_b.$$

Consider now the case in which neither V nor h have compact support. Pointwise, we have:

$$\langle L_b^*[V], h\rangle - \langle V, L_b[h]\rangle = \langle Hess(V), e\rangle - (\Delta V)tr_b h - V\delta(\delta h) + V\Delta(tr_b h).$$

$$-\delta(\delta h) + \Delta(tr_b h) = \delta\mu, \quad \mu = -\delta h - d(tr_b h) \in \Omega^1_M.$$

In particular, if b is Ricci-flat, the constants (say $V \equiv 1$) are in $Ker L_b^*$, and we have, integrating over the compact manifold with boundary M:

$$\int_M L_b[h]d\mu_b = -\int_M (\delta\mu)d\mu_b = \int_M div_b(\mu^\#) = \int_{\partial M} \mu[\nu_b]d\sigma_b,$$

by the divergence theorem (ν_b is the unit outward normal in the background metric). This relates the bulk integral of linearized scalar curvature to the boundary integral of the ADM mass integrand. $\mu[\nu_b]$

In the general case, we have:

$$\delta[V(-\delta h - d(tr_b h)) - i_{\nabla V} h + (tr_b h) dV] =$$

$$\langle dV, \delta h\rangle - V\delta(\delta h) + \langle dV, d(tr_b h)\rangle - V\delta d(tr_b h) - \delta(i_{\nabla V} h) - \langle d(tr_b h), dV\rangle - (tr_b h)\Delta V.$$

Now use:

$$\delta(i_{\nabla V} h) = (\delta h)(\nabla V) - \langle Hess(V), h\rangle.$$

After cancelation, we find the pointwise relation:

$$\delta(\mu_{(V,h)}) = \langle L_b^*[V], h\rangle - \langle V, L_b[h]\rangle, \quad \mu_{(V,h)} = V(-\delta h - d(tr_b h)) - i_{\nabla V} h + (tr_b h) dV \in \Omega^1_M.$$

Integrating over the compact manifold with boundary M

$$\int_M (\langle V, L_b[h]\rangle - \langle L_b^*[V], h\rangle) d\mu_b = \int_M div_b(\mu^{\#}_{(V,h)}) d\mu_b = \int_{\partial M} \mu_{(V,h)}[\nu_b] d\sigma_b.$$

3. Variation of the Einstein-Hilbert functional.

We consider the first variation of:

$$\mathcal{R}_b = \int_M S_b d\mu_b$$

under the variation of metric: $g^t = b + th$, $h \in Sym^2_M$. We have:

$$\dot{S} = -\Delta(tr_b h) + \delta(\delta h) - \langle Ric, h\rangle_b, \quad (\dot{d\mu}) = \frac{1}{2}(tr_b h) d\mu_b.$$

Thus:

$$\dot{\mathcal{R}} = \int_M [\delta(\delta h + d(tr_b h)) - \langle G^b, h\rangle] d\mu_b, \quad G^b = Ric^b - \frac{S_b}{2} b,$$

the 'Einstein tensor' of b. Using the divergence theorem:

$$\dot{\mathcal{R}} = -\int_M \langle G^b, h\rangle d\mu_b + \int_{\partial M} \mu_h[\nu_b] d\sigma_b, \quad \mu_h = -\delta h - d(tr_b h) \in \Omega^1_M.$$

We conclude the critical metrics for $\mathcal{R}$ (under variations h with compact support) are those with vanishing Einstein tensor G. And for metrics with vanishing G and variation tensor $h = \dot{g}$, we have the suggestive relation:

$$\dot{\mathcal{R}} - \dot{m} = 0, \quad m = \int_{\partial M} \mu_g[\nu_b] d\sigma_b.$$

4. Where the mass comes from. (cp. [Michel] and [Herzlich].)

We consider (M,g) complete noncompact (with one end, for simplicity), asymptotic to a 'background' $(\mathbb{M}_0, b)$ (typically euclidean or hyperbolic n-space, in the following sense: there exists a chart from an exterior region (complement of a ball) $E_0 \subset \mathbb{M}_0$ to $M \setminus K$, $K \subset M$ compact: $\phi : E_0 \to M \setminus K$, and letting $g_\phi = \phi^* g$, we have on E_0:

$$h = g_\phi - b = O_2(r^{-\tau}),$$

where r is distance in $\mathbb{M}_0$ to a fixed point. Consider the space of 'static potentials' on b:

$$\mathcal{N}_b = \{V \in C^\infty(\mathbb{M}_0; L_b^*[V] = 0\}, \quad L_b^*(V) = Hess(V) - (\Delta_b V)b - V Ric_b,$$

the formal adjoint of the linearization at b of the scalar curvature map. Consider the Taylor theorem representation:

$$Scal(g_\phi) - Scal(b) = L_b[h] + Q_b(h),$$

where $Q_b(h)$ is the quadratic (and higher order) remainder.

Exercise. If $V \in \mathcal{N}_b$, the metric $V^2 dt \oplus b$ on $\mathbb{R} \times \mathbb{M}_0$ is Ricci-flat.

The main assumption is g_ϕ is asymptotic to b as $r \to \infty$, at a fast enough rate that:
(1) $\langle V, Scal(g_\phi) - Scal(b)\rangle_b$ is $d\mu_b$-integrable;
(2) $Q_b(V,h) = \langle V, Q_b(h)\rangle_b$ is $d\mu_b$-integrable, for all $V \in \mathcal{N}_b$.

If $V \in \mathcal{N}_b$, pointwise on E_0 we have:

$$\begin{aligned}\langle V, Scal(g_\phi) - Scal(b)\rangle &= \langle V, L_b[h]\rangle + \langle V, Q_b(h)\rangle \\ &= -\delta_b(\mu^\phi(V,h)) + \langle V, Q_b(h)\rangle_b,\end{aligned}$$

for a one-form $\mu^\phi(V,h)$ on E_0 given in the previous section (linear in V, h and their first order derivatives.)

Let (B_k) be an increasing exhaustion of $\mathbb{M}_0$ by Smoothly bounded domains, $\partial B_k = S_k$ (so for $k \geq k_1$ large enough, $S_k \subset E_0$). The divergence theorem implies:

$$\oint_{S_k} \mu^\phi(V,h)[\nu_b] d\sigma_b = \oint_{S_{k_1}} \mu^\phi(V,h)[\nu_n] d\sigma_b + \int_{B_k \setminus B_{k_1}} [\langle V, Scal(g_\phi - Scal(b)\rangle - Q_b(V,h)] d\mu_b.$$

The integrability conditions (1) and (2) guarantee the RHS has a limit as $k \to \infty$, and that this limit is independent of k_1. We conclude the limit:

$$\lim_k \oint_{S_k} \mu^\phi(V,h)[\nu_b] d\sigma_b$$

exists, and is independent of the exhaustion considered.

Remark: Additional conditions are needed to ensure the limit is independent of the chart ϕ. Roughly speaking, they are: (i) $Scal(b)$ is constant and (ii) Any two charts ϕ, ψ with the property that g_ϕ, g_ψ are asymptotic to b differ by a diffeomorphism (of some exterior region E_R) whose leading term is an isometry of b. (This is a 'rigidity condition' on b; for details, see [Michel].)

Example 1: asymptotically flat manifolds.
$(\mathbb{M}_0, b) = (\mathbb{R}^n, \delta)$: euclidean space.

We assume $h = g - \delta = O_2(r^{-\tau})$ on E_0. The remainder term satisfies:

$$|Q(1,e)| \leq C(|\partial h|^2 + |h|_b|\partial^2 h|) = O(r^{-2\tau-2}).$$

Thus the integrablity conditions (1), (2) above are satisfied if $Scal_g \in L^1(M,g)$ and $2\tau + 2 > n$, or $\tau > \frac{n-2}{2}$.

Since $L_0^*[V] = Hess_0(V) - (\Delta_0 V)\delta$ (euclidean Hessian and Laplacian), taking traces one sees easily that $V \in \mathcal{N}_b$ iff $Hess_0(V) = 0$, or V is affine (linear plus constant): $\mathcal{N}_b$ is $(n+1)$-dimensional. For $V = 1$, the mass one-form is:

$$\mu(1,h) = \sum_{i,j} (\partial_i h_{ij} - \partial_j h_{ii}) dx^j \in \Omega^1(E_0),$$

and the mass is:

$$m(g) = \lim_k \oint \mu(1,h)[\nu_0] d\sigma_0,$$

up to a normaliziation constant depending only on n.

Example 2: asymptotically hyperbolic manifolds. $(\mathbb{M}_0, b) = (\mathbb{H}^n, g_{hyp})$, hyperbolic space.: $g_{hyp} = dr^2 + \sinh^2 r g_{S^{n-1}}$, the standard metric on the sphere.The asymptotic conditions are:

$$|h| = O_2(e^{-r\tau}), \quad h = \phi^* g - b.$$

The static potentials V (kernel of L_b^*) are solutions of:

$$Hess_b(V) - (\Delta_b V)b = V(Ric_b) = -(n-1)Vb.$$

It is easy to see (*exercise*) this is equivalent to $Hess_b(V) = Vb$. As in the euclidean case, the space $\mathcal{N}_b$ of solutions has dimension $n+1$:

$$\mathcal{N}_b = span\{V_0, V_1, \ldots, V_n\}, \quad V_0 = \cosh r, V_i = (\sinh r)\omega_i,$$

where $\omega = (\omega_1, \ldots, \omega_n)$ is the standard embedding of S^{n-1} into R^n.

Consider the integrability conditions (1) and (2): since $|V| + |\nabla V| = O(e^r)$, we need $e^r(Scal(\phi^* g) + n(n-1))$ integrable; while $|Q_b(V,h)| = O(e^r e^{-2\tau r})$; since $d\mu_b = O(e^{(n-1)r})$, we have $Q_b(V,h)d\mu_b = O(e^{nr}e^{-2\tau r})$, so for integrability we need: $\tau > n/2$. Under these conditions, the limit:

$$m_g(V) = \lim_k \oint_{S_k} \mu_b(V,h)[\nu_b] d\sigma_n, \quad \mu_b(V,h) = V(-\delta_b h - d(tr_b h)) - i_{\nabla^b V} h + (tr_b h) dV \in \Omega^1(E_0)$$

exists, and defines a linear functional on $\mathcal{N}_b$.