Sliced spacetimes. Let (V^{n+1}, g) be a Lorentzian manifold with signature $(- + \ldots +)$, diffeomorphic to $M^n \times R$. We assume V carries a time function t, meaning the integral curves of the vector field ∂_t, dual to the differential dt (i.e. $dt[\partial_t] \equiv 1$) are positive-oriented timelike, and foliate V. The level sets M_t of t are spacelike n-manifolds, diffeomorphic to M. Let n be the future-directed timelike unit vector field normal to the level sets M_t. (t, x_i) are adapted local coordinates, with associated vector fields (∂_t, ∂_i), with $\partial_i \in TM_t$. We have the decomposition:

$$\partial_t = Nn + X, \quad X = X^i \partial_i, \qquad n = \frac{1}{N}(\partial_t - X)$$

for a function $N = -g(n, \partial_t) > 0$ in V (the 'lapse function') and vector field $X \in TM_t$ (the 'shift vector field'), $|X| < N$. (Note N and X are uniquely determined by the time function t and the metric g.) Denoting by D and ∇ the Levi-Civita connections for g and for the induced metric γ on M_t (resp.), we define the second fundamental form K of M_t via:

$$D_Y W = \nabla_Y W + K(Y, W)n, \quad K(Y, W) = -\gamma(D_Y W, n) = \gamma(D_Y n, W), \quad Y, W \in TM_t.$$

To find the expression for the metric g in the frame $\{\partial_t, \partial_i\}$, we compute:

$$g(\partial_t, \partial_t) = -N^2 + |X|^2, \quad g(\partial_k, \partial_t) = X^j \gamma_{jk}, \quad g(\partial_k, \partial_l) = \gamma_{kl}.$$

This gives, recalling $dx^k dt = (1/2)(dx^k \otimes dt + dt \otimes dx^k)$:

$$\begin{aligned} g &= (-N^2 + |X|^2)dt^2 + 2X^j \gamma_{jk} dx^k dt + \gamma_{ij} dx^i dx^j \\ &= -N^2 dt^2 + \gamma_{ij}(dx^i + X^i dt)(dx^j + X^j dt). \end{aligned}$$

We compute the covariant derivative $D_n n$:

$$\langle D_n n, \partial_j \rangle = -\langle n, D_n \partial_j \rangle = -\langle n, D_{\partial_j} n \rangle + \langle n, [\partial_j, n] \rangle = \langle n, [\partial_j, n] \rangle,$$

$$[\partial_j, n] = -\frac{\partial_j N}{N} n - \frac{1}{N}[\partial_j, X],$$

$$\langle D_n n, \partial_j \rangle = -\frac{\partial_j N}{N} \langle n, n \rangle = \frac{\partial_j N}{N},$$

hence:

$$D_n n = \gamma^{ij} \frac{\partial_i N}{N} \partial_j = \frac{1}{N} \nabla^\gamma N.$$

For a tangent vector field $Z \in TM_t$, define the operator:

$$\mathcal{L}_0 Z = [\partial_t, Z] - [X, Z].$$

Note that if $Z = b^i\partial_i, [\partial_t, Z] = (\partial_t b^j)\partial_j \in TM_t$, so $\mathcal{L}_0 Z \in TM_t$.

This implies:

$$[n, Z] = [\frac{1}{N}(\partial_t - X), Z] = \frac{1}{N}\mathcal{L}_0 Z - Z(\frac{1}{N})(\partial_t - X) = \frac{1}{N}\mathcal{L}_0 Z + \frac{Z(N)}{N}n.$$

Decomposition of the curvature tensor. Let $Y, Z \in TM_t$ be vector fields on V.

$$\langle D_Y D_n n, Z\rangle = \langle \nabla_Y(\frac{\nabla N}{N}), Z\rangle = \frac{1}{N}\nabla^2 N(Y, Z) - \frac{Y(N)Z(N)}{N^2}.$$

$$\begin{aligned}\langle D_n D_Y n, Z\rangle &= n\langle D_Y n, Z\rangle - \langle D_Y n, D_n Z\rangle \\ &= n(K(Y, Z)) - \langle D_Y n, D_Z n\rangle - \langle D_Y n, [n, Z]\rangle \\ &= n(K(Y, Z)) - K^2(Y, Z) - \frac{1}{N}K(Y, \mathcal{L}_0 Z).\end{aligned}$$

$$\begin{aligned}\langle D_{[Y,n]} n, Z\rangle &= -\frac{Y(N)}{N}\langle D_n n, Z\rangle - \frac{1}{N}\langle D_{\mathcal{L}_0 Y} n, Z\rangle \\ &= -\frac{Y(N)Z(N)}{N^2} - \frac{1}{N}K(\mathcal{L}_0 Y, Z).\end{aligned}$$

Thus we have the curvature component:

$$R^V(Y, n, n, Z) = \langle D_Y D_n n - D_n D_Y n - D_{[Y,n]} n, Z\rangle$$

$$= \frac{1}{N}\nabla^2 N(Y, Z) + K^2(Y, Z) - n(K(Y, Z)) + \frac{1}{N}K(Y, \mathcal{L}_0 Z) + \frac{1}{N}K(\mathcal{L}_0 Y, Z)$$

$$= \frac{1}{N}\nabla^2 N(Y, Z) + K^2(Y, Z) - \frac{1}{N}(\mathcal{L}_0 K)(Y, Z),$$

where we define:

$$(\mathcal{L}_0 K)(Y, Z) = \partial_t(K(Y, Z)) - X(K(Y, Z)) - K(\mathcal{L}_0 Y, Z) - K(Y, \mathcal{L}_0 Z).$$

We also have the Gauss and Codazzi equations (where $X, Y, Z, W \in TM_t$):

$$R^V(X, Y, Z, W) = R^M(X, Y, Z, W) + K(X, W)K(Y, Z) - K(Y, W)K(X, Z),$$

$$R^V(X, Y, n, Z) = (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z).$$

The constraint equations. Define $Q(Y,Z) = R^V(Y,n,n,Z)$ (symmetric). Then $Rc^V(n,n) = tr_\gamma Q$, while:

$$Rc^V(X,Y) = \sum_i R^V(X,e_i,e_i,Y) - R^V(X,n,n,Y),$$

hence from the Gauss equation:

$$Rc^V(X,Y) = Rc^M(X,Y) + HK(X,Y) - K^2(X,Y) - Q(X,Y), \quad X,Y \in TM,$$

where $H = tr_\gamma K$. For the scalar curvature:

$$R^V = -Rc^V(n,n) + \sum_j Rc^V(e_j,e_j)$$
$$= R^M + H^2 - |K|^2 - 2tr_\gamma Q.$$

For the Einstein tensor, the terms involving Q cancel, and we obtain:

$$G^V(n,n) = Rc^V(n,n) - \frac{1}{2}R^V g(n,n) = Rc^V(n,n) + \frac{R^V}{2}$$
$$= \frac{1}{2}(R^M + H^2 - |K|^2).$$

So the Einstein vacuum equation $G^V = 0$ gives the 'Hamiltonian constraint':

$$R^M + H^2 - |K|^2 = 0 \text{ on } M.$$

The 'momentum constraint' follows from:

$$G^V(X,n) = \sum_i (\nabla_{e_i}K)(e_i,X) - X(H), \quad X \in TM.$$

The evolution equations.

(i) For the metric on M:

$$\partial_t\gamma = \mathcal{L}_X\gamma + 2NK, \text{ or } \quad \mathcal{L}_0\gamma = 2NK.$$

Proof.

$$\mathcal{L}_X\gamma(\partial_i,\partial_j) = \gamma(\nabla_{\partial i}X,\partial_j) + \gamma(\nabla_{\partial_j}X,\partial_i)$$
$$= g(D_{\partial t}\partial_i,\partial_j) + g(D_{\partial_t}\partial_j,\partial_i) - N[g(D_{\partial_i}n,\partial_j) + g(D_{\partial_j}n,\partial_j)]$$
$$= \partial_t\gamma_{ij} - 2NK_{ij}.$$

(ii) For the second fundamental form:

$$\mathcal{L}_0 K = \nabla^2 N + N(K^2 - Q), \text{ where } Q = -Rc^V + Rc^M + HK - K^2 \in Sym_2(TM).$$

Proof. Follows directly from what was proved above:

$$Q = N^{-1}\nabla^2 N + K^2 - N^{-1}\mathcal{L}_0 K.$$

Einstein metrics and static metrics.

The Einstein metric condition $Rc^V = \lambda g$ is equivalent to two:

$$Rc^V_{|TM} = \lambda\gamma, \quad Rc^V(n,n) = tr_\gamma Q = -\lambda,$$

where the second one can also written as:

$$-\lambda = N^{-1}\Delta N + |K|^2 - N^{-1} tr\mathcal{L}_0 K,$$

or equivalently:

$$\Delta N + \lambda N = tr\mathcal{L}_0 K - N|K|^2.$$

The first one can be written in the form:

$$Rc^M + HK - K^2 - \lambda\gamma = N^{-1}\nabla^2 N + K^2 - N^{-1}\mathcal{L}_0 K.$$

Taking traces, we find:

$$R^M - n\lambda = N^{-1}\Delta N + 2|K|^2 - H^2 - N^{-1} tr\mathcal{L}_0 K.$$

Comparing the two scalar equations, we find for the scalar curvature:

$$R^M = (n-1)\lambda + |K|^2 - H^2.$$

The Ricci curvature of M is given by:

$$Rc^M - \lambda\gamma = N^{-1}\nabla^2 N + 2K^2 - HK - N^{-1}\mathcal{L}_0 K.$$

If $K \equiv 0$ on V (product case, with $g = -N^2 dt^2 \oplus \gamma$), this simplifies to:

$$Rc^M - \lambda\gamma = N^{-1}\nabla^2 N, \quad \Delta N + \lambda N = 0.$$

(So N is a 'static potential' on M.) Solving for λ we have in the product case:

$$Rc^M = N^{-1}(\nabla^2 N - (\Delta N)\gamma).$$

In the general case:

$$Rc^M = N^{-1}(\nabla^2 N - (\Delta N)\gamma) + 2K^2 - HK - |K|^2\gamma - N^{-1}(\mathcal{L}_0 K - (tr\mathcal{L}_0 K)\gamma).$$