ENERGY-MOMENTUM: spherically symmetric initial data sets [1]

**1. Spherically symmetric Positive Mass Theorem.**

We consider metrics in $M = \mathbb{R}^n \setminus B_R$ of the form:

$$g = \frac{dr^2}{V(r)} + r^2 d\omega^2, \quad (r,\omega) \in [R,\infty) \times S^{n-1}.$$

Geometrically, $r \geq R$ is the "area radius", in the sense that the $(n-1)$-dimensional area of the sphere $\{r\} \times S^{n-1} = S_r$ is:

$$|S_r|_g = \omega_{n-1} r^{n-1}.$$

Since $\nu = \sqrt{V}\partial_r$ is the outward unit normal of $S_r$, the first variation formula gives for the mean curvature $H(r)$ of $S_r$::

$$\omega_{n-1} H(r) r^{n-1} = \sqrt{V}\partial_r |S_r|_g = \sqrt{V}\frac{n-1}{r}\omega_{n-1} r^{n-1}, \text{ or } H(r) = \frac{n-1}{r}\sqrt{V}.$$

The second fundamental form of $S_r$ is $A = \frac{1}{r}\sqrt{V}\mathbb{I}_{n-1}$.

We compute the scalar curvature $R^M$ of $M$ in the metric $g$. Recall the formula for the differential (first variation) of the mean curvature of a hypersurface $\Sigma$, with respect to a normal variation with velocity field $\phi\nu$:

$$DH[\phi\nu] = -\Delta_\Sigma \phi + \frac{1}{2}(R^\Sigma - R^M - |A|^2 - H^2)\phi.$$

And with $\phi \equiv 1$:

$$DH[\nu] = \frac{1}{2}(R^\Sigma - R^M - |A|^2 - H^2).$$

In the spherically symmetric case, for $\Sigma = S_r$:

$$|A|^2 = \frac{n-1}{r}^2, \quad H^2 = \frac{(n-1)^2}{r^2}V, \quad R^{S_r} = \frac{(n-1)(n-2)}{r^2},$$

and also:

$$\nu(H) = \sqrt{V}\partial_r H = (n-1)\sqrt{V}\partial_r(\frac{\sqrt{V}}{r}) = (n-1)[-\frac{V}{r^2} + \frac{V'}{2r}].$$

This gives:

$$R^M = -2\nu(H) + R^{S-r} - |A|^2 - H^2 = \frac{n-1}{r^2}[(n-2)(1-V) - rV'].$$

[1] Course notes, based on [D. Lee, pp. 251-254], and including some computations omitted there.

Under the assumption $R^M \geq 0$, this implies a monotonicity relation:

$$\frac{d}{dr}[r^{n-2}(1-V)] = r^{n-3}[(n-2)(1-V) - rV'] = \frac{r^{n-1}}{n-1}R^M \geq 0.$$

*Remark.* If $n = 3$, $r^{n-2}(1-V)$ is the *Hawking mass* of $S_r$, up to a constant depending on $n$.

*ADM mass.* Assume $g$ is asymptotically flat; in particular, $V(r) \to 1$ as $r \to \infty$. Recall the definition:

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_r \int_{S_r} (g_{ij,j} - g_{jj,i})\nu_0^i d\sigma, \quad \nu_0^i = \frac{x^i}{r}.$$

(With implicit summation over repeated indices.) We compute this for rotationally symmetric $g$. Writing $g = \delta + e$, where $\delta$ is the euclidean metric:

$$e = (\frac{1}{V} - 1)dr^2, \quad tr_0 e = \frac{1}{V} - 1.$$

Thus we have:

$$\sum_i x^i \partial_i(tr_0 e) = r\partial_r(tr_0 e) = r\partial_r(\frac{1}{V}).$$

For the other term, consider:

$$\sum_{i,j} x^i \partial_j e_{ij} = \sum [\partial_j(x^i e_{ij}) - \delta_{ij}e_{ij} = \sum [\partial_j(e(x^i\partial_i, \partial_j)) - (\frac{1}{V} - 1)$$

$$= \sum_j \partial_j(e(r\partial_r, \partial_j)) - (\frac{1}{V} - 1).$$

Since

$$e(r\partial_r, \partial_j) = \langle \partial_j, \partial_r \rangle_0 e(r\partial_r, \partial_r) = \frac{x^j}{r} r(\frac{1}{V} - 1) = x^j(\frac{1}{V} - 1),$$

we have:

$$\sum_j \partial_j(e(r\partial_r, \partial_j)) = n(\frac{1}{V} - 1) + r\partial_r(\frac{1}{V}).$$

Thus the mass integrand is given by:

$$\sum_{i,j} x^i\partial_j e_{ij} - x^i\partial_i(tr_0 e) = n(\frac{1}{V}-1) + r\partial_r(\frac{1}{V}) - (\frac{1}{V}-1) - r\partial_r(\frac{1}{V}) = (n-1)(\frac{1}{V}-1).$$

We conclude:

$$m = \lim_r \frac{1}{2(n-1)\omega_{n-1}}(\omega_{n-1}r^{n-1})\frac{n-1}{r}(\frac{1}{V}-1) = \lim_r \frac{1}{2}r^{n-2}(\frac{1}{V}-1) = \lim_r \frac{1}{2}r^{n-2}(1-V),$$

since $V(r) \to 1$ as $r \to \infty$.

In the case $R = 0$, so $M = \mathbb{R}^n$ with the spherically symmetric $g$, necessarily $V(r) \to 1$ as $r \to 0_+$. Thus if the scalar curvature $R^M \geq 0$, the monotone function $m(r) = \frac{1}{2} r^{n-2}(1 - V)$ increases from 0 to $m$, and we have $m \geq 0$ (and it's easy to see that $m = 0$ only if $g$ is the euclidean metric)–the *Riemannian positive mass theorem* in the spherically symmetric case.

For $R > 0$, and assuming $S_R$ is minimal for the rotationally symmetric metric $g$, we know $V(R) = 0$, and thus $m(r)$ increases from $\frac{1}{2} R^{n-2} = \frac{1}{2} (\frac{|S_R|_g}{\omega_{n-1}})^{\frac{n-2}{n-1}}$ to $m$, and we conclude:

$$m \geq \frac{1}{2} (\frac{|S_R|_g}{\omega_{n-1}})^{\frac{n-2}{n-1}},$$

the *Riemannian Penrose inequality* in the spherically symmetric case.

**2. Spherically symmetric initial data sets.**

We consider initial data sets $(M, g, k)$, where $M = \mathbb{R}^n \setminus B_R = [R, \infty) \times S^{n-1}$ and:

$$g = \frac{dr^2}{V} + r^2 d\omega^2, \quad k = \frac{k_0}{V} dr^2 + \frac{\kappa}{n-1} r^2 d\omega^2$$

with $V > 0, k_0, \kappa$ functions of $r$. We see that, with $\nu = \sqrt{V} \partial_r$: $k_0 = k(\nu, \nu), \kappa = tr_{g|S_r} k = \sum_i k(e_i, e_i)$, with $(e_i) \in TS_r$ $g$-orthonormal. The asymptotic flatness conditions assumed on $(g, k)$ are:

$$V = 1 + O_2(r^{-q}), \quad k_0 = O_1(r^{-q-1}), \quad \kappa = O_1(r^{-q-1}), \text{ as } r \to \infty, \text{ with } q > \frac{n-2}{2}.$$

The ADM energy-momentum vector of $(g, k)$ is defined as:

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_r \int_{S_r} (g_{ij,j} - g_{jj,i}) \nu_0^i d\sigma, \quad P_i = \frac{1}{(n-1)\omega_{n-1}} \lim_r \int_{S_r} (k_{ij} - (tr_g k) g_{ij}) \nu_0^j d\sigma,$$

with summation convention.

It is easy to see that the ADM momentum $(P_i)$ vanishes, in the spherically symmetric case:

$$k(\partial_i, \nu_0) - (tr_g k) \frac{x^i}{r} = k_0 \frac{x^i}{r} - (k_0 + \kappa) \frac{x^i}{r} = -\kappa \frac{x^i}{r},$$

while $\int_{S_r} x^i r^{n-1} d\sigma(x) = 0$ for all $r$.

We now compute expressions for the energy density $\mu$ and normal component of the current density, $\langle J, \nu \rangle_g$. First, since $(tr_g k)^2 = (k_0 + \kappa)^2$ and $|k|_g^2 = k_0^2 + \frac{\kappa^2}{n-1}$,

$$\mu := \frac{1}{2}[R^M + (tr_g k)^2 - |k|_g^2] = \frac{1}{2}[R^M + 2\kappa k_0 + \frac{n-2}{n-1}\kappa^2].$$

Turning now to the current density,

$$\langle J, \nu \rangle_g = (div_g k)(\nu) - \nu(tr_g k),$$

where, with $e_0 = \nu$ and $(e_i) \in TS_r$ orthonormal,

$$(div_g k)(\nu) = \sum_{i\geq 0}(\nabla_{e_i} k)(e_i, \nu) \sum_{i\geq 0}[e_i(k(e_i, \nu)) - k(e_i, \nabla_{e_i}\nu) - k(\nabla_{e_i} e_i, \nu\rangle$$

$$= \sqrt{V}\partial_r k_0 - \sum_{i\geq 1} k(e_i, \nabla_{e_i}\nu) - \sum_{i\geq 1}\langle \nabla_{e_i} e_i, \nu\rangle k_0,$$

where in the last two sums the term $i = 0$ vanishes since $\nabla_{e_0}\nu \perp \nu$ and $k(e_0, e_i) = 0$. We have, in view of the sign convention adopted for the second fundamental form of $S_r$:

$$\sum_{i\geq 1} k(e_i, \nabla_{e_i}\nu) = \sum_{i,j\geq 1} k(e_i, e_j)\langle e_j, \nabla_{e_i}\nu\rangle = \sum_i k(e_i, e_i)A(e_i, e_i) = \frac{\kappa}{n-1}H$$

Combining this with:

$$\nu(tr_g k) = \sqrt{V}\partial_r(k_0 + \kappa),$$

we find:

$$\langle J, \nu\rangle = \sqrt{V}\partial_r k_0 - \frac{\kappa}{n-1}H + k_0 H - \sqrt{V}\partial_r(k_0 + \kappa)$$

$$= H(-\frac{\partial_r \kappa}{n-1} + k_0 - \frac{\kappa}{n-1}),$$

where we used $\sqrt{V} = \frac{rH}{n-1}$ (cf. [Lee, Exercise 7.47]).

**3. Dominant energy condition and monotonicity.**

Combining the above we find, where $H > 0$:

$$\mu - \frac{\kappa}{h}\langle J, \nu\rangle = \frac{R^M}{2} + \frac{1}{n-1}[r\kappa\partial_r\kappa + \frac{n\kappa^2}{2}] = \frac{R^M}{2} + \frac{r^{-(n-1)}}{2(n-1)}\partial_r(r^n\kappa^2).$$

Now recall from part (1) above:

$$\frac{1}{2}R^M = (n-1)r^{-(n-1)}\partial_r[\frac{1}{2}r^{n-2}(1-V)].$$

This motivates defining:

$$m_k(r) := \frac{1}{2}r^{n-2}(1-V) + \frac{r^n\kappa^2}{2(n-1)^2}.$$

And we have:

$$\partial_r m_k = \frac{r^{n-1}}{n-1}[\mu - \frac{\kappa}{H}\langle J, \nu\rangle].$$

(cp. the Claim in [Lee, p.252].)

Since $V = r^2H^2/(n-1)^2$, we have the alternative expression:

$$m_k(r) = \frac{1}{2}r^{n-2}[1 + \frac{r^2}{(n-1)^2}(\kappa^2 - H^2)].$$

Recall the inward and outward expansion scalars for $S_R$ are defined as:

$$\theta_- = \kappa - H, \quad \theta_+ = \kappa + H.$$

Thus, if $S_R$ is either a MOTS ($\theta_+ = 0$) or a MITS ($\theta_- = 0$), we have $\kappa^2 = H^2$, so $m_k(R) = \frac{1}{2}R^{n-2} = \frac{1}{2}(\frac{|S_R|_g}{\omega_{n-1}})^{\frac{n-2}{n-1}}$.

On the other hand, recall $\kappa^2 = O(r^{-2q-2})$ with $q > \frac{n-2}{2}$, or $2q + 2 > n$. So $\kappa^2 r^n \to 0$ at infinity, and from the first expression for $m_k$ we conclude $m_k(\infty) = \lim_r \frac{1}{2}r^{n-2}(1 - V(r)) = E$.

Since $H = \frac{n-1}{r}\sqrt{V} = \frac{n-1}{r}(1 + O(r^{-q}))$ and $\kappa = O(r^{-q-1})$, we have $|\kappa| < H$ at infinity. This implies $|\kappa| \leq H$ for all $r > R$, *assuming* $S_R$ is an outermost MOTS or MITS (otherwise there would be some $r > R$ for which $\theta_+\theta_- = 0$, contradicting 'outermost'.) Now recall the DEC $\mu \geq |J|$ to see that:

$$\mu - \frac{\kappa}{H}\langle J, \nu\rangle \geq \mu - \frac{|\kappa|}{H}|J| \geq \mu - |J| \geq 0,$$

which implies from the calculation in (2) that $m_k(r)$ is monotone nondecreasing. Thus we have the *Penrose inequality for spherically symmetric data sets*:

**Theorem.** Assume $(M, g, k)$ (spherically symmetric) satisfies the DEC $\mu \geq |J|$ and that $S_R$ is an outermost MOTS or MITS. Then we have the lower bound:

$$E - |P| \geq \frac{1}{2}(\frac{|S_R|_g}{\omega_{n-1}})^{\frac{n-2}{n-1}}.$$

**4. The case of equality: embedding spherically symmetric metrics into Schwarzschild spacetime.**

We wish to show that if equality holds in the Penrose inequality for spherically symmetric data sets $(M, g, k)$, then $(M, g)$ can be isometrically embedded as a spacelike hypersurface in Schwarzschild spacetime $\mathcal{S}_m$ with mass parameter $m$, with $k$ as its second fundamental form. Write the Schwarzschild metric as:

$$g_m = -V_m dt^2 + V_m^{-1}dr^2 + r^2 d\omega^2, \quad V_m = 1 - \frac{2m}{r^{n-2}}, \quad r > r_m = (2m)^{\frac{1}{n-2}}.$$

Let $f(r), r \geq r_0$ be a radial function on $M = \mathbb{R}^n \setminus B_{r_0}$. Use $f$ to embed $M$ as a spacelike hypersurface in $\mathcal{S}_m$, the graph of $f$ (we need $|f'| < V_m^{-1}$ for this.) Denote the embedding by:

$$F : M \to \mathcal{M}_f \subset \mathcal{S}_m, \quad F(r, \omega) = (f(r), r, \omega),$$

that is, $t = f(r)$. Pulled back to $M$, the induced metric is spherically symmetric, of the form:

$$g_f = \frac{dr^2}{V_f} + r^2 d\omega^2, \quad V_f^{-1} = V_m^{-1} - V_m(f')^2.$$

To see this, consider:

$$V_f^{-1} = \langle \partial_r, \partial_r \rangle_{g_f} = \langle (f'(r)\partial_t, \partial_r), (f'(r)\partial_t, \partial_r) \rangle_{g_m} = -V_m(f')^2 + V_m^{-1},$$

and this will be positive (as needed for a spacelike embedding) if $|f'| < V_m^{-1}$.

Next we consider the second fundamental form $k_f$ of the graph, and **claim** that, pulled back to $M$, it has the form:

$$k^f = k_0^f \frac{dr^2}{V_f} + \frac{\kappa^f}{n-1} d\omega^2, \text{ where } k_0^f = \frac{d}{dr}\sqrt{V_f - V_m}, \quad \kappa^f = \frac{n-1}{r}\sqrt{V_f - V_m}.$$

(We already know $V_f \geq V_m$.)

*Proof of the claim.* We first compute the unit timelike normal $n$ to the graph $\mathcal{M}_f$. Let $\eta = (\partial_t, f' V_m^2 \partial_r) \in T\mathcal{S}_m$. With $F_* \partial_r = f'\partial_t + \partial_r$, we have:

$$\langle \eta, F_*\partial_r = \langle \eta, f'\partial_t + \partial_r \rangle_{g_m} = -V_m f' + V_m^{-1} V_m^2 f' = 0,$$

$$\langle \eta, \eta \rangle_{g_m} = -V_m + V_m^{-1}(f')^2 V_m^4 = V_m(-1 + V_m^2 (f')^2) < 0,$$

So $\eta$ is normal to the graph $\mathcal{M}_f$, and future-timelike. To get a unit vector, set $w^2 = (1 - V_m^2(f')^2)V_m$, and let $n = \frac{\eta}{w}$. Then $g_m(n, n) = -1$.

We compute the second fundamental form of the graph $\mathcal{M}_f$, pulled back to $M$. First note that:

$$V_f = \frac{V_m}{1 - V_m^2(f')^2} \Rightarrow V_f - V_m = \frac{V_m^3 (f')^2}{1 - V_m^2(f')^2} \Rightarrow \sqrt{V_f - V_m} = \frac{V_m^{3/2} f'}{\sqrt{1 - V_m^2(f')^2}} = \frac{V_m^2 f'}{w}.$$

To simplify the notation, let $h = w^{-1} V_m^2 f' = \sqrt{V_f - V_m}$. Then:

$$k^f(\partial_r, \partial_r) = \langle \nabla_{F_*\partial_r}(\frac{\eta}{w}), F_*\partial_r \rangle_{g_m} = a + b + c + d,$$

and compute using:

$$\langle \partial_t, \partial_t \rangle = -V_m, \quad \langle \partial_t, \partial_r \rangle = 0, \quad \langle \partial_r, \partial_r \rangle = V_m^{-1}, \quad \nabla_{\partial_t}\partial_r = \nabla_{\partial_r}\partial_t$$

(where $\nabla$ denotes the Levi-Civita connection for $g_m$ in $\mathcal{S}_m$) to find:

$$a = (f')^2 [w^{-1} \langle \nabla_{\partial_t}\partial_t, \partial_t \rangle + \langle \nabla_{\partial_t}(h\partial_r), \partial_t \rangle] = -\frac{1}{2}(f')^2 h V_m',$$

$$b = f' \langle \nabla_{\partial_r}(w^{-1}\partial_t + h\partial_r, ), \partial_t \rangle - f'(\frac{w'}{w^2} V_m - \frac{1}{2w} V_m'),$$

$$c = f' \langle \nabla_{\partial_t}(w^{-1}\partial_t + h\partial_r), \partial_r \rangle = \frac{f' V_m'}{2w},$$

$$d = \langle \nabla_{\partial_r}(w^{-1}\partial_t + h\partial_r, \partial_r \rangle = \frac{h'}{V_m} + \frac{h}{2}(V_m^{-1})'.$$

Thus:

$$a+b+c+d = -\frac{1}{2}(f')^2 V_m' h + f'\frac{w'}{w^2}V_m + h'V_m^{-1} - \frac{h}{2}\frac{V_m'}{V_m^2}, \qquad (*)$$

and the claim is that this equals:

$$V_f^{-1}(\sqrt{V_f - V_m})' = h'(V_m^{-1} - V_m(f')^2).$$

We compute:

$$h' = \frac{1}{2h}(V_f' - V_m') = \frac{1}{2h}[(\frac{V_m^2}{w^2})' - V_m'] = \frac{w}{2V_m^2 f'}[\frac{2V_m V_m'}{w^2} - \frac{2V_m^2 w'}{w^2} - V_m']$$

$$= \frac{V_m'}{V_m w f'} - \frac{w'}{w^2 f'} - \frac{w V_m'}{2V_m^2 f'},$$

and thus:

$$h'(V_m^{-1} - V_m(f')^2) = -\frac{V_m' f'}{w} + \frac{w' f' V_m}{w^2} + \frac{w V_m' f'}{2V_m} + h'V_m^{-1}, \qquad (**)$$

and we see two terms from (*) are already matched. As for the remaining terms, on the one hand, in (*):

$$-\frac{1}{2}V_m'(f')^2 h - \frac{hV_m'}{2V_m^2} = -\frac{h}{2}V_m'((f')^2 + \frac{1}{V_m^2}) = -\frac{f'V_m'}{2w}(V_m^2(f')^2 + 1),$$

and on the other, in (**):

$$-\frac{V_m' f'}{w} + \frac{wV_m' f'}{2V_m} = -\frac{V_m' f'}{2w}(2 - \frac{w^2}{V_m}) = -\frac{V_m' f'}{2w}(1 + V_m^2(f')^2),$$

and we see that (*) and (**) are indeed equal.

With $\nu = \sqrt{V_f}\partial_r$ the outward unit vector to $S_r$, we then have:

$$k_0^f = k^f(\nu,\nu) = V_f k^f(\partial_r,\partial_r) = (\frac{f'V_m^2}{w})' = (\sqrt{V_f - V_m})',$$

so we have established the first part of the claim, regarding $k^f(\nu,\nu)$,

Turning now to $\kappa^f$, let $e \in TS_r$ be a unit vector. Then, using $F^*n = w^{-1}V_m^2 f'\partial_r$:

$$\frac{\kappa^f}{n-1} = k^f(e,e) = \langle \nabla_e^{S_r}(F^*n), e\rangle_{g_f} = -\langle F^*n, \nabla_e^{S_r} e\rangle_{S_r} = \langle F^*n, \frac{1}{r}\partial_r\rangle_{S_r} = \frac{1}{r}\frac{f'V_m^2}{w}.$$

Thus:

$$\kappa^f = \frac{n-1}{r}\frac{V_m^2 f'}{w} = \frac{n-1}{r}\sqrt{V_f - V_m},$$

as claimed.