OSCILLATION OF SECTIONAL CURVATURE AND IRREDUCIBLE COMPONENTS OF THE RIEMANN TENSOR

Recall the irreducible O(n)-orthogonal decomposition of the (4,0) algebraic Riemann curvature tensor, for $n \geq 4$ [Besse, p. 48]:

$$R = W + \frac{1}{n-2} Ric^\circ \owedge g + \frac{scalar}{2n(n-1)} g \owedge g,$$

where $\owedge$ is the Kulkarni-Nomizu product of two symmetric 2-forms:

$$(h \owedge k)(x, y, z, t) = h(x, z)k(y, t) - h(y, z)k(x, t) + h(y, t)k(x, z) - h(x, t)k(y, z).$$

We show here that the Weyl and Ric° components, as well as all entries of R except for sectional curvatures, can be bounded by a dimensional constant times the sectional curvature oscillation, $\overline{\kappa} - \underline{\kappa}$.

Let $\overline{\kappa}, \underline{\kappa}$ be the maximum and minimum sectional curvatures at a point (over all two-planes.) Given an orthonormal frame (e_i), denote by $K_{ij} = sect\{e_i, e_j\}$, the sectional curvature of the 2-plane spanned by e_i, e_j. We have for the scalar curvature:

$$scalar = 2\sum_{j>1} K_{1j} + 2\sum_{1<i<j} K_{ij},$$

and for the trace-free Ricci curvature Ric°:

$$\begin{aligned} Ric^\circ(e_1, e_1) &= \sum_{j>1} K_{1j} - \frac{2}{n}\sum_{j>1} K_{1j} - \frac{2}{n}\sum_{1<i<j} K_{ij} \\ &= \frac{n-2}{n}\sum_{j>1} K_{1j} - \frac{2}{n}\sum_{1<i<j} K_{ij} \\ &\leq \frac{n-2}{n}(n-1)\overline{\kappa} - \frac{2}{n}\frac{(n-1)(n-2)}{2}\underline{\kappa} = \frac{(n-1)(n-2)}{n}(\overline{\kappa} - \underline{\kappa}). \end{aligned}$$

Thus:

$$|Ric^\circ| \leq \frac{(n-1)(n-2)}{n}(\overline{\kappa} - \underline{\kappa}).$$

We now consider bounds on the Riemann tensor entries. For entries with four different vectors, we have Berger's bound, see [Bre, p.5] for a proof:

$$|R(e_1, e_2, e_3, e_4)| \leq \frac{2}{3}(\overline{\kappa} - \underline{\kappa}).$$

One sees easily that $(Ric^\circ \owedge g)(e_1, e_2, e_3, e_4) = 0$ and $(g \owedge g)(e_1, e_2, e_3, e_4) = 0$, and hence also for the Weyl tensor:

$$|W(e_1, e_2, e_3, e_4)| \leq \frac{2}{3}(\overline{\kappa} - \underline{\kappa}).$$

Consider now the calculation:

$$2sect\{e_1, e_2+e_3\} = R(e_1, e_2+e_3, e_1, e_2+e_3) = R(e_1, e_2, e_1, e_2)+R(e_1, e_3, e_1, e_3)+2R(e_1, e_2, e_1, e_3).$$

It implies:

$$R(e_1, e_2, e_1, e_3) = sect\{e_1, e_2 + e_3\} - \frac{1}{2}sect\{e_1, e_2\} - \frac{1}{2}sect\{e_1, e_3\},$$

and thus:

$$|R(e_1, e_2, e_1, e_3)| \leq \overline{\kappa} - \underline{\kappa}.$$

One also easily checks the decomposition:

$$R(e_1, e_2, e_1, e_3) = W(e_1, e_2, e_1, e_3) + \frac{1}{n-2}Ric^\circ(e_2, e_3),$$

from which the bound on the Weyl tensor follows:

$$|W(e_1, e_2, e_1, e_3)| \leq (1 + \frac{n-1}{n})(\overline{\kappa} - \underline{\kappa}).$$

For the sectional curvature-type entries, we obviously have:

$$R(e_1, e_2, e_1, e_2) = sect\{e_1, e_2\} \leq \overline{\kappa}.$$

For the Weyl tensor, consider:

$$W(e_1, e_2, e_1, e_2) = sect\{e_1, e_2\} - \frac{1}{n-2}(Ric^\circ(e_1, e_1) + Ric^\circ(e_2, e_2)) - \frac{scalar}{n(n-1)}.$$

Thus (since $scalar/n(n-1)$ is an average of sectional curvatures):

$$W(e_1, e_2, e_1, e_2) \leq \overline{\kappa} - \frac{2}{n-2}\frac{(n-1)(n-2)}{n}(\overline{\kappa} - \underline{\kappa}) - \underline{\kappa},$$

or:

$$|W(e_1, e_2, e_1, e_2)| \leq (1 + \frac{2(n-1)}{n})(\overline{\kappa} - \underline{\kappa}).$$

We conclude that, for a constant $c_n > 0$ depending only on dimension:

$$|W| + \frac{1}{n-2}|Ric^0| + \sum\{|R(e_i, e_j, e_k, e_l)|;\ \text{three or four different indices}\}| \leq c_n(\overline{\kappa} - \underline{\kappa}).$$

while, for each $i \neq j$:

$$sect\{e_i, e_j\} \leq \overline{\kappa} \leq (\overline{\kappa} - \underline{\kappa}) + \underline{\kappa} \leq (\overline{\kappa} - \underline{\kappa}) + \frac{scalar}{n(n-1)}.$$

In summary:

$$|R - \frac{scalar}{n(n-1)}g \owedge g| \leq c_n(\overline{\kappa} - \underline{\kappa}) \leq c_n(\frac{\overline{\kappa}}{\underline{\kappa}} - 1)\underline{\kappa} \leq \frac{c_n}{n(n-1)}(\frac{\overline{\kappa}}{\underline{\kappa}} - 1)scalar.$$

Discussion/ exercises.

1. For $n \geq 4$, we have an orthogonal decomposition of the space of algebraic curvature tensors of an inner-product space (V, g):

$$\mathcal{C}_B = \mathcal{W} \oplus \mathcal{R} \oplus \mathcal{S},$$

where $\mathcal{S} = \mathbb{R} g \owedge g$, $\mathcal{W} = Ker(Rc)$, the kernel of the Ricci contraction map:

$$Rc : \mathcal{C}_B \to Sym_2 V, \quad Rc(R)(x, y) = \sum_i R(x, e_i, y, e_i).$$

Remark: $Rc(h \owedge g) = (n-2)h + (tr_g h)g$.

Also $\mathcal{R} = Ker(\tau)$, the kernel of the double trace map:

$$\tau : \mathcal{C}_B \to \mathbb{R}, \quad \tau(R) = \sum_{i,j} R(e_i, e_j, e_i, e_j).$$

Vanishing of each of the components of R has the usual geometric meaning: $W = 0$ ($\mathcal{W}$ component vanishes) characterizes locally conformally flat metrics, $Ric^0 = 0$ ($\mathcal{R}$ component vanishes) characterizes Einstein metrics.

Finally, recall that $\mathcal{C}_B$ is the kernel of the Bianchi map from $\mathcal{C} = Sym_2(\Lambda^2_V)$ to itself ($b^2 = b$):

$$b(R)(x, y, z, t) = \frac{1}{3}\{R(x, y, z, t) + R(z, x, y, t) + R(y, z, x, t)\}.$$

(i) Compute the dimensions of the vector spaces and subspaces defined here, when $n \geq 4$.

(ii) Describe the corresponding decomposition when $n = 3$.

2. As established in [Bre, p.53], the Hamilton vector field $Q(R) = R^2 + R^\#$ maps $\mathcal{C}_B$ to $\mathcal{C}_B$. Thus the following question is natural: *how does Q behave with respect to the orthogonal decomposition of $\mathcal{C}_B$?* Are any of the subspaces $\mathcal{W}$, $\mathcal{R}$, $\mathcal{S}$ invariant under Q? (Note that invariance of $\mathcal{W} \oplus \mathcal{S}$ (resp. $\mathcal{W} \oplus \mathcal{R}$, resp. $\mathcal{R} \oplus \mathcal{S}$) means, via Hamilton's maximum principle, Ricci flow preserves the class of Einstein metrics (resp. constant curvature metrics, resp. conformally flat metrics).