Calculus Review

1. $\{a_n\}$, $\lim_{n\to\infty} a_n$, $\sum_{n=0}^{\infty} a_n$

2. Intermediate Value Thm: If $f \in \mathcal{C}[a,b]$, then f takes on every value between $f(a), f(b)$.

3. derivative

4. integral

5. Taylor expansion (approximation by polynomials) about a point:

If f has $n+1$ derivatives at x_0, $f(x)$ can be approximated near x_0 by the n-th degree Taylor polynomial: $f(x) \approx P_n(x)$ near x_0

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= a_0 + a_1 \cdot (x-x_0) + a_2 \cdot (x-x_0)^2 + \dots + a_n \cdot (x-x_0)^n$$

The coefficients $a_k = \frac{f^{(k)}(x_0)}{k!}$ are chosen for derivatives at x_0 to match:

$$P_n^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, 1, \dots, n$$

The error in the approximation $f(x) \approx P_n(x)$ is called the remainder $R_n(x, x_0)$

$$f(x) = P_n(x) + R_n(x, x_0) \quad \text{for } x \text{ near } x_0$$

Lagrange form of the Remainder: If $f \in \mathcal{C}^n(a,b)$ and $f^{(n+1)}(x)$ exists in (a,b)

$$R_n(x, x_0) \equiv f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \quad \text{for some } \xi \text{ between } x \text{ and } x_0$$

Thus, the error reduces for smaller $x - x_0$, or for larger n

$$R_n = \mathcal{O}(\Delta x^{n+1}) \quad \text{as } \Delta x = x - x_0 \to 0$$

Note that in R_n, $f^{(n+1)}(\xi)$ is evaluated at ξ, not at x_0, for some ξ btw x and x_0.

Thus, to estimate $|R_n|$ need an upper bound on $|f^{(n+1)}(x)|$, $x \in (a,b)$

6. Alternating series: $\sum_0^{\infty} (-1)^n a_n$: if $a_n \searrow 0$ as $n \to \infty$ then the alt. series converges and the error of stopping at N-th partial sum is $\le a_{N+1}$

Important Taylor series expansions

1. $e^x = \underbrace{\sum_{k=0}^{n} \frac{x^k}{k!}}_{P_n(x)} + \underbrace{R_n(x,0)}_{\frac{e^\xi}{(n+1)!}x^{n+1}}$ about $x_0=0$ $= 1 + x + \frac{x^2}{2!} + \dots$

   with ξ btn 0 and x

   As $n \to \infty$, $\sum_0^\infty \frac{x^k}{k!}$ converges to e^x $\forall x$ (∞ radius of convergence by Ratio Test)

2. $\sin x = \sum_0^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \forall x$

3. $\cos x = \sum_0^\infty (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad \forall x$

4. $\ln(1+x) = \sum_1^\infty (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad -1 < x \le 1$

5. $\frac{1}{1-x}$ = sum of geometric series $= \sum_0^\infty x^k = 1 + x + x^2 + \dots \quad |x| < 1$

6. Harmonic series $\sum_1^\infty \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges (slowly)

7. Alternating harmonic $\sum_1^\infty \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges! slowly $= \ln 2$

Example: How many terms needed to find $\cos(0.1)$ to 8 decimals?

$\cos(0.1) \approx P_{2k}(0.1)$ with error $|R_{2k}(0.1,0)| = \frac{|\sin/\cos\ \xi|}{(2k+1)!}(0.1)^{2k+1} \le \frac{1}{(2k+1)!} \cdot 10^{-(2k+1)}$

want $\le \frac{1}{2} \cdot 10^{-8}$

Try $k=2$: $\frac{10^{-5}}{5!} \overset{?}{\le} \frac{10^{-8}}{2}$, $2 \cdot 10^3 \overset{?}{\le} 5!$ no

$k=3$: $\frac{10^{-7}}{7!} \overset{?}{\le} \frac{10^{-8}}{2}$, $2 \cdot 10 \overset{?}{\le} 7!$ yes, by a lot!

So $\cos(0.1) \approx 1 - \frac{(.1)^2}{2!} + \frac{(.1)^4}{4!} - \frac{(.1)^6}{6!} = 0.9950041 = P_6(0.1)$

How many correct digits? $|\text{error}| \le \frac{(.1)^7}{7!} = \frac{10^{-7}}{5040} = .1984 \cdot 10^{-10} < \frac{1}{2} 10^{-10}$

so 10 digits, at least

Example: How many terms for $\cos(1)$?

$|\text{error}| \le \frac{1^{2k+1}}{(2k+1)!} \overset{\text{want}}{\le} \frac{1}{2} 10^{-8}$, $\quad 2 \cdot 10^8 \overset{\text{want}}{\le} (2k+1)!$

$k=5$ too small, $k=6$ works, so need $P_{12}(1)$, because $x - x_0 = 1 - 0 = 1$ is ~~not~~ small