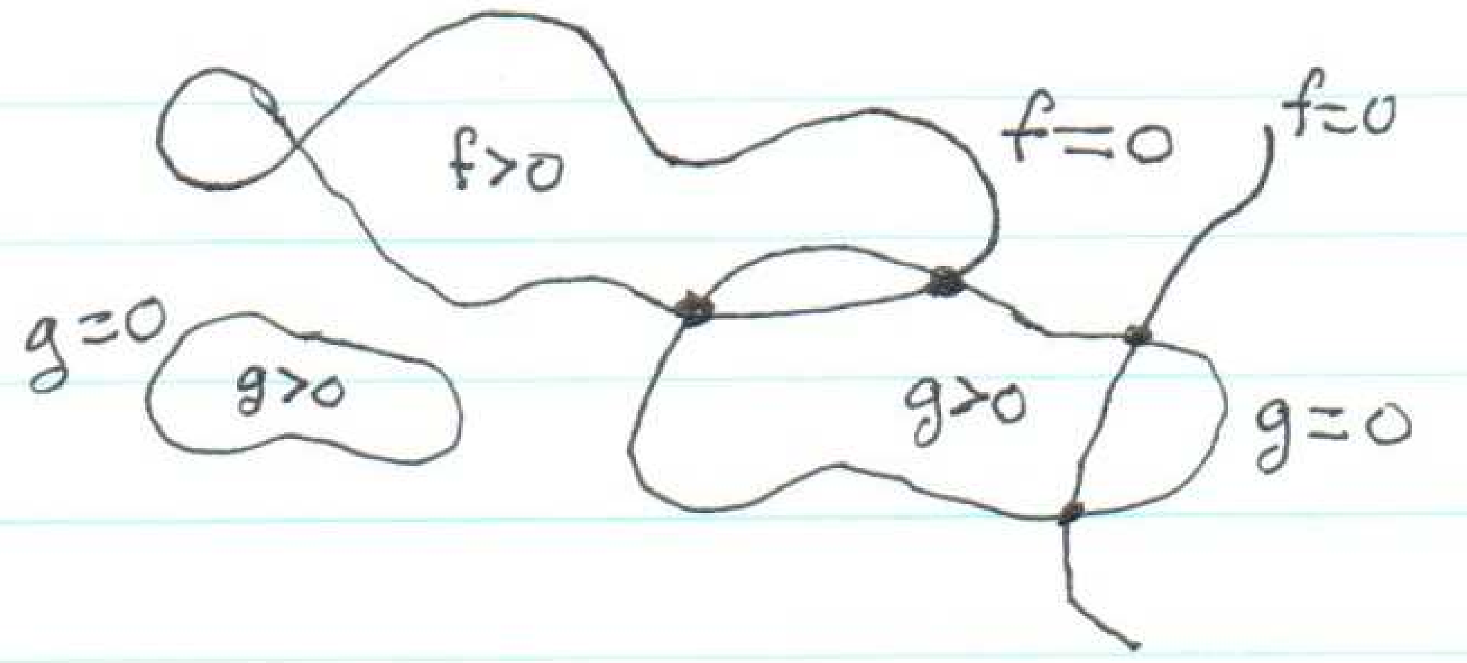
Nonlinear systems

$\vec{F}(\vec{x}) = \vec{0}$ N (nonlinear) equations in N unknowns

$$\begin{cases} F_1(x_1, x_2, \dots, x_N) = 0 \\ F_2(\qquad\qquad) = 0 \\ \dots \\ F_N(\qquad\qquad) = 0 \end{cases}$$

It is a <u>hard</u> math problem! do sols exist? No good general methods...

<u>2×2 system:</u> $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ find intersections of two curves



Bisection loses its meaning

Fixed point iteration, Newton-Raphson may be applicable

<u>Newton-Raphson</u> via Taylor expansion:

Let $P_k(x_k, y_k)$ be the current iterate. Expand about P_k:

$$\begin{cases} 0 \overset{\text{want}}{=} f(x_k + \Delta x,\ y_k + \Delta y) = f(x_k, y_k) + \frac{\partial f}{\partial x}(P_k)\cdot\Delta x + \frac{\partial f}{\partial y}(P_k)\cdot \Delta y + O(\Delta x^2, \Delta y^2) \\ 0 \overset{\text{want}}{=} g(\qquad\qquad) = g(P_k) + \frac{\partial g}{\partial x}(P_k)\cdot \Delta x + \frac{\partial g}{\partial y}(P_k)\cdot\Delta y + \dots \end{cases}$$

find increments $\Delta x, \Delta y$:

$$\Rightarrow \begin{cases} \left.\frac{\partial f}{\partial x}\right|_{P_k}\cdot \Delta x + \left.\frac{\partial f}{\partial y}\right|_{P_k}\cdot\Delta y = -f(P_k) \\ \left.\frac{\partial g}{\partial x}\right|_{P_k}\cdot\Delta x + \left.\frac{\partial g}{\partial y}\right|_{P_k}\cdot\Delta y = -g(P_k) \end{cases}$$

2×2 <u>linear</u> system for $\Delta x, \Delta y$!

In matrix form: $\vec{\Delta x} = \begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix}$, $J_k = \left.\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\right|_{P_k}$ = Jacobian of $\vec{F}(\vec{x}) = \begin{bmatrix} f \\ g\end{bmatrix}$

solve $J_k \vec{\Delta x} = -F_k$ for $\vec{\Delta x}$ corresponds to $f'(x_k)\Delta x = -f(x_k)$

then $\vec{x}_{k+1} = \vec{x}_k + \vec{\Delta x}$, $k = 0, 1, \dots$ till convergence

Newton step: $\vec{\Delta x} = -J_k^{-1} \vec{F}_k$ corresponds to $\Delta x = -\frac{f_k}{f'_k}$ in 1-D

found by numerically solving the linear system $J_k \vec{\Delta x} = -F_k$ by some method
(never by inverting J_k !) direct or iterative

The role of f' in 1-D is played by the <u>Jacobian matrix</u> J in N-D

$$J = [J_{ij}] = \left[\frac{\partial f_i}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & & \vdots \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

Has all the advantages and disadvantages of 1-D case

Very expensive: $\sim N^2$ ops to ~~compute~~ evaluate Jacobian, much more for complicated f_i
$\sim N^3$ to solve the linear system

Example: $\begin{cases} x_1(1-x_1) + 4x_2 = 12 \\ (x_1-2)^2 + (2x_2-3)^2 = 25 \end{cases}$ parabola $y = 3 - \frac{x(1-x)}{4}$
circle at $(2, \frac{3}{2})$ of radius 5

$$J = \begin{bmatrix} 1-x_1-x_1 & 4 \\ 2(x_1-2) & 2(2x_2-3)\cdot 2 \end{bmatrix} = \begin{bmatrix} 1-2x_1 & 4 \\ 2(x_1-2) & 4(2x_2-3) \end{bmatrix}$$

Initial guess: $X^0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow F^0 = \begin{bmatrix} -2+12-12 \\ 0+9-25 \end{bmatrix} = \begin{bmatrix} -2 \\ -16 \end{bmatrix}$, $J^0 = \begin{bmatrix} -3 & 4 \\ 0 & 12 \end{bmatrix}$

Solve $\begin{bmatrix} -3 & 4 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = -\begin{bmatrix} -2 \\ -16 \end{bmatrix} \Rightarrow \Delta x_2 = \frac{4}{3} \Rightarrow -3\Delta x_1 + 4 \cdot \frac{4}{3} = 2 \Rightarrow \Delta x_1 = \frac{10}{9}$

Then $X^1 = X^0 + \vec{\Delta x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 10/9 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 28/9 \\ 13/3 \end{bmatrix}$

Repeat until $\|\vec{\Delta x}\| < TOL$ and $\|F\| < TOL$