Errors in $\boxed{\text{polynomial}}$ interpolation

We view interpolation as approximation of an underlying function $f(x)$ whose values generate the given data points.

<u>Theorem</u>: Let $P_N(x)$ be the polynomial of $\deg \le N$ that interpolates $f(x)$ at $N+1$ distinct nodes $x_0, x_1, \ldots, x_N$ in $[a,b]$.
If $f \in C^{N+1}[a,b]$ (i.e. $f, f', \ldots, f^{(N+1)}$ are continuous on $[a,b]$)
then for each $x \in [a,b]$ $\exists\, \xi_x \in (a,b)$ such that

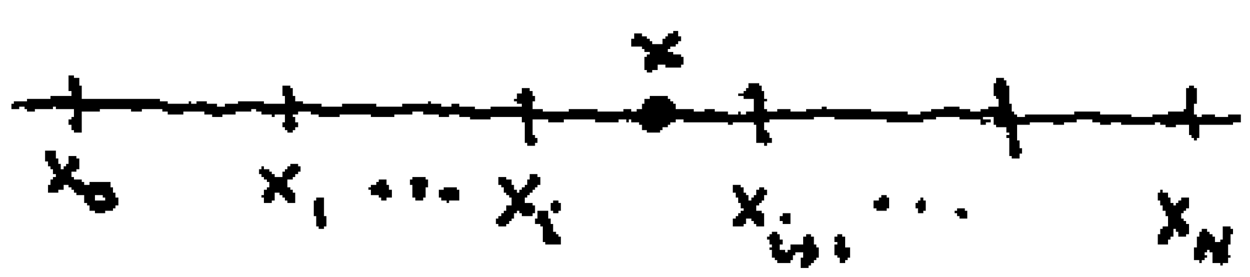
$$f(x) - P_N(x) = \frac{f^{(N+1)}(\xi_x)}{(N+1)!}(x-x_0)(x-x_1)\cdots(x-x_N)$$

[Note similarity with Remainder of Taylor]

Hence, if $h = \max_i |\Delta x_i|$ and $\|f^{(N+1)}\|_\infty \le M$ on $[a,b]$ then the <u>approximation error</u> is

$$|f(x) - P_N(x)| \le \frac{M}{4(N+1)} \cdot h^{N+1} \quad \forall x \in [a,b]$$

Proof of estimate: $|(x-x_0)\cdots(x-x_N)| \le \frac{h}{2}\cdot\frac{h}{2}\cdot 2h \cdot 3h \cdots Nh = \frac{N!}{4}\cdot h^{N+1}$



so

$$|\text{error}| \le \frac{M}{N!\,(N+1)}\cdot\frac{N!}{4}\cdot h^{N+1} = \frac{M}{4(N+1)}\cdot h^{N+1}$$

<u>Note</u>: The error can be big (whenever $h > 1$ and N is big, even for moderate M).
Smells bad for high degree poly. interpolation, unless $h \ll 1$ but often we have no choice on the nodes (available measurements).

<u>Roundoff error</u> (for equispaced nodes): Nodes $x_i = a + i \cdot h$, $h = \frac{b-a}{N}$, $i = 0, \dots, N$

where we are given values $\tilde{f}_i = f(x_i) + \varepsilon_i$, ε_i = roundoff / measurement error

Poly. interpolation of exact values would produce the poly. $P_N(x)$

" " " given " " " " " $\tilde{P}_N(x)$

Then, max roundoff error (due to data errors)

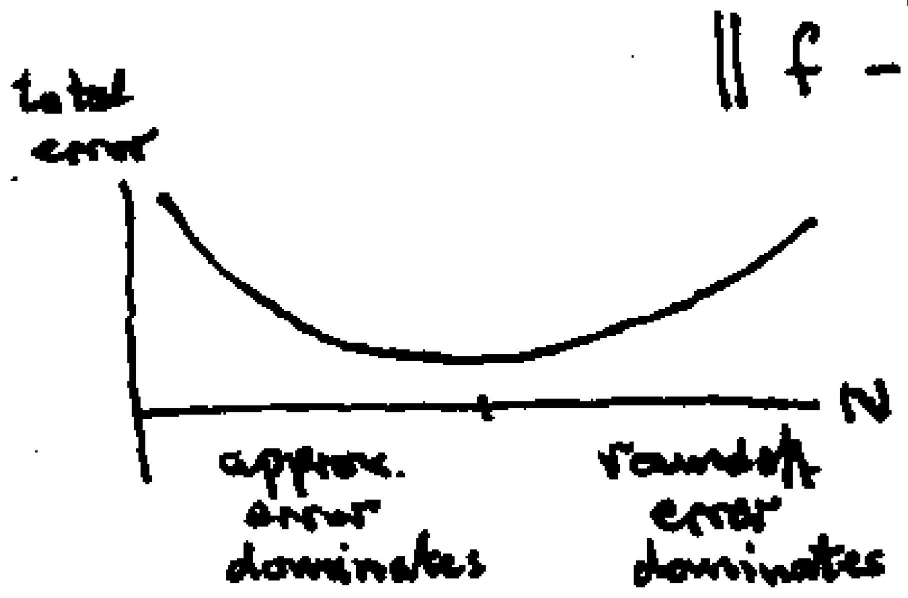
$$\| P_N - \tilde{P}_N \|_\infty \leq \frac{2^N + 1}{2} \cdot \varepsilon \quad , \quad \varepsilon = \max_i \varepsilon_i = \text{max data error}$$

The number $\kappa = \frac{2^N+1}{2}$ is the amplification factor on data errors

may be thought of as <u>condition number for polynomial interpolation</u>.

We see the danger of using large N: makes it ill-conditioned.

<u>Total error in polynomial interpolation</u>: $f - \tilde{P}_N = f - P_N + P_N - \tilde{P}_N$, so



$$\| f - \tilde{P}_N \|_\infty \leq \underbrace{\| f - P_N \|_\infty}_{\text{approx. error}} + \underbrace{\| P_N - \tilde{P}_N \|_\infty}_{\text{roundoff error}}$$

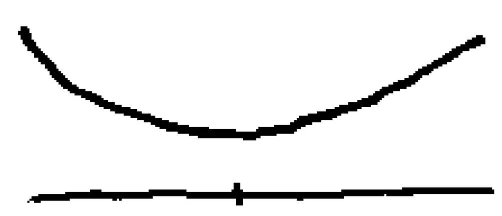
$$\leq \frac{\| f^{(N+1)} \|_\infty}{4(N+1)} \cdot h^{N+1} + \frac{2^N+1}{2} \cdot \varepsilon$$

Even if we can choose dense nodes to make $h \ll 1$,

as N grows: approx. error may $\searrow$ but roundoff error $\nearrow$.

Situation is similar to Finite Difference approximation of derivatives, or worse...

Such behavior occurs in every discretization process:



total error = discretization error + roundoff error

Moral of the story: Avoid high degree polynomial interpolation!

Use piecewise interpolation of <u>low</u> degree ($N = 2$ or 3)

over every 3 or 4 nodes

And, whenever possible, choose dense nodes (small h).