Numerical Solution of ODEs

Basic Concepts

1. DE = Differential Equation: an equation for an unknown function involving the unkown and its derivatives

e.g. $y' + y - e^t = 1$: $y = y(t)$
$3x\,y'' + y^2 = 0$; $y = y(x)$
$(t^2+1)\,y'' - 3yy' + 6t = 0$: $y = y(t)$

$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$; $u = u(x,t)$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$; $u = u(x,y)$

Order of a DE is the order of highest derivative appearing in the DE.

ODE : DE for unknown of one independent variable: $y(t)$ M231, 431, 531,
PDE : DE for unknown of many indep. variables: $u(x,y,t)$ M435, 475, 535-6, 635-

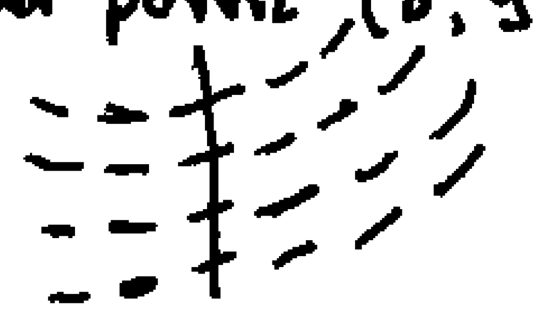
Solution of a DE is a function that satifies the DE at each t

e.g. $y' = y + e^t$ has a solution $y(t) = Ce^t + te^t$ for any const. C
indeed: $y' = [Ce^t + te^t]' = Ce^t + e^t + te^t \equiv y + e^t \quad \forall C$

Some simple types of ODEs can be solved analytically (M231, 431-2)
all others only numerically!
Solution of n-th order ODE involves n arbitrary constants!

In general, a given DE has many solutions (or none)
so additional conditions must be specified in order to determine a single solution

e.g. $y' = y + e^t$ only specifies the slope $y'(t)$ at each point (t,y),
a direction field on (t,y) plane;



but ∞-ly many curves have such slopes
To pick one trajectory, must specify a "starting" point (t_0, y_0)
i.e. $y(t_0) = y_0$

2. We will consider only 1^{st} order ODEs in standard form: $y' = f(t,y)$
with an initial condition: $y(t_0) = y_0$. Constitutes an Initial Value Problem

standard IVP for 1^{st} order ODE : (IVP) $\begin{cases} y' = f(t,y)\,, & t_0 \le t \le t_{end} \quad \text{ODE} \\ y(t_0) = y_0 & \text{IC} \end{cases}$

$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ is equivalent to the Integral Equ: $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$

3. Well-posedness

A problem is <u>well-posed</u> if
1. solution(s) exist (Existence property)
2. solution is unique (Uniqueness)
3. solution depends continuously on data (Continuous dep. on data)
(small changes in data $\Rightarrow$ small change in solution)

else it is called <u>ill-posed</u>.

Inverse problems are ill-posed! (given a solution find the problem)
most important: parameter identification problems are ill-posed!
(from measurements of solution find physical properties)

<u>Well-posedness Theorem for</u> (IVP) $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

If $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are <u>continuous</u> and <u>bounded</u> on a rectangle $R = [a,b] \times [c,d]$ of (t,y) plane; then for any $(t_0, y_0) \in R$ the IVP is well-posed near the initial pt (t_0, y_0)

i.e. there <u>exists</u> <u>unique</u> solution $y(t)$ in some subinterval $(t_0 - r,\ t_0 + r), r>0$ which depends continuously on the data.

If $\frac{\partial f}{\partial y} >> 0$ then trajectories get further apart as $t \uparrow$, errors get amplified, ill-conditioned (IVP), equilibria are unstable

If $\frac{\partial f}{\partial y} < 0$ then trajectories get closer, errors are damped as $t \uparrow$, well-conditioned (IVP), equilibria are stable

Hard to tell which case because sign of $\frac{\partial f}{\partial y}$ depends on y, may be changing...

Typically, to test num. methods we apply them to prototype linear IVP: $\begin{cases} y' = \lambda y \\ y(0) = y_0 \end{cases}$
whose solution is $y(t) = y_0 e^{\lambda t}$

4. Systems of 1^{st} order:

$$(\text{IVP})\begin{cases} y_1' = f_1(t, y_1, y_2, \cdots, y_n) & , \; y_1(t_0) = y_1^\circ \\ y_2' = f_2(t, y_1, \cdots, y_n) & , \; y_2(t_0) = y_2^\circ \\ \vdots & \vdots \\ y_n' = f_n(t, y_1, \cdots, y_n) & , \; y_n(t_0) = y_n^\circ \end{cases}$$

can write it in vector form:

$$(\text{IVP})\begin{cases} \vec{y}\,' = \vec{F}(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}^\circ \end{cases} \text{ where } \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \vec{F} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \vec{y}^\circ = \begin{bmatrix} y_1^\circ \\ \vdots \\ y_n^\circ \end{bmatrix}$$

All methods developed for scalar case (single ODE) also apply to vector case and all ODE solver software are for systems.

5. Higher order IVPs can be written as 1^{st} order systems:

$$(\text{IVP})\begin{cases} y^{(n)} = g(t, y, y', \dots, y^{(n-1)}) \\ y(t_0) = y_0 , \; y'(t_0) = y_1^\circ , \dots , \; y^{(n-1)}(t_0) = y_{n-1}^\circ \end{cases}$$

set $w_1 = y$, $w_2 = y'$, $\dots$, $w_n = y^{(n-1)}$, then

$$(\text{IVP})\begin{cases} w_1' = w_2 & , \; w_1(t_0) = y_0 \\ w_2' = w_3 & , \; w_2(t_0) = y_1^\circ \\ \vdots & \\ w_{n-1}' = w_n & , \; w_{n-1}(t_0) = y_{n-2}^\circ \\ w_n' = g(t, w_1, \dots, w_n) & , \; w_n(t_0) = y_{n-1}^\circ \end{cases} \qquad \text{i.e.} \begin{cases} \vec{w}\,' = \vec{F}(t, \vec{w}) \\ \vec{w}(t_0) = \vec{w}^\circ = \vec{y}^\circ \end{cases}$$

e.g. $3xy'' + y^2 = 0$: set $w_1 = y$, $w_2 = y'$ $\Rightarrow \begin{cases} w_1' = w_2 \\ w_2' = y'' = -\dfrac{y^2}{3x} = -\dfrac{w_1^2}{3x} \end{cases}$

or $\vec{w}\,' = \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} w_2 \\ -\dfrac{w_1^2}{3x} \end{bmatrix} =: \vec{F}(t, \vec{w})$ $\quad 2 \times 2$ system

Numerical solutions for (IVP) $\begin{cases} y' = f(t,y), & t_0 < t < t_{end} \\ y(t_0) = y_0 \end{cases}$

By the Well-posedness Thm, if f, $\frac{\partial f}{\partial y}$ are cont's and bounded on some R, then the IVP is well-posed (at least locally).

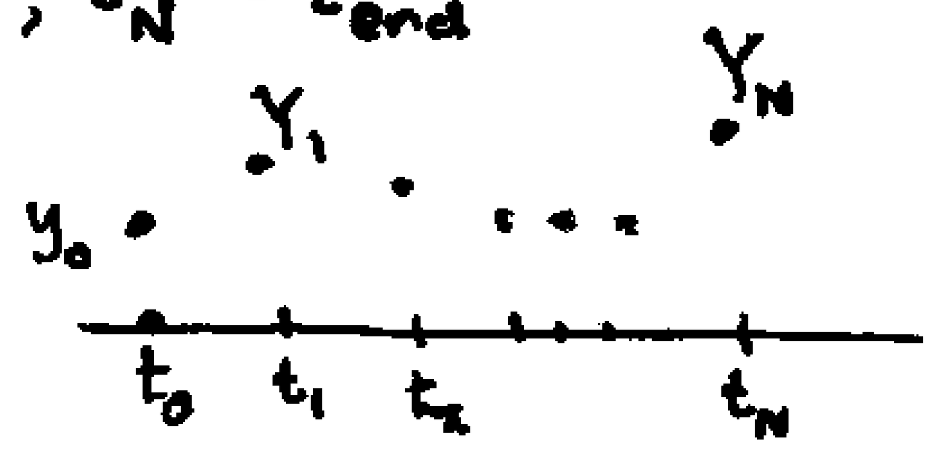
Exact analytic solution are possible only for very particular $f(t,y)$, studied in M231. All others must be approximated numerically.

Numerical solution by time-discretization:

Discetize time interval $[t_0, t_{end}]$ into N subintervals by (usually) equispaced nodes $t_0 < t_1, \dots, t_N = t_{end}$

with time-steps $\Delta t = \frac{t_{end} - t_0}{N} = h$

$$t_n = t_0 + n \cdot \Delta t \;, \quad n = 0:N$$



and seek approximations

$$Y_n \approx y(t_n) \;, n = 0:N \quad \text{only at the N times } t_n$$

with $Y_0 = y_0 = y(t_0)$ from IC.

Numerical schemes produce values $Y_0, Y_1, \dots, Y_n, \dots, Y_N$ by <u>time-stepping</u>.

discretization error $= y(t_n) - Y_n$ should $\to 0$ as $\Delta t \to 0$ (more later...)

There are <u>many</u> schemes, and more are being developed still...

Dictum: There is <u>no</u> best method for all problems!

Numerical methods should be:

consistent, convergent, stable (more later)

and efficient and robust

Computational cost: number of $f(t_n, Y_n)$ evaluations needed to get to t_{end}