<u>Least Squares in $L^2(a,b)$</u> or in $L^2_w(a,b)$

Approximate $f \in L^2(a,b)$ by $\Phi(x,\vec{c}) := \sum_{k=1}^{n} c_k \varphi_k(x)$, basis functions $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$:

Find coefficients $\vec{c}$ to minimize $E(\vec{c}) = \| f - \Phi \|^2_{L^2} \equiv \int_a^b |f(x) - \Phi(x,\vec{c})|^2 dx$

The normal equations are: $\left(A\vec{c} \overset{L^2}{=} f \right)$ $n \times n$ linear system

$$\begin{bmatrix} \langle \varphi_1, \varphi_1 \rangle & \langle \varphi_1, \varphi_2 \rangle & \cdots & \langle \varphi_1, \varphi_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle \varphi_n, \varphi_1 \rangle & \langle \varphi_n, \varphi_2 \rangle & \cdots & \langle \varphi_n, \varphi_n \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \varphi_1 \rangle \\ \langle f, \varphi_2 \rangle \\ \vdots \\ \langle f, \varphi_n \rangle \end{bmatrix}$$

If we chose <u>orthonormal</u> set of basis functions then $\langle \varphi_k, \varphi_\ell \rangle = \delta_{k\ell} = \begin{cases} 0, & k \neq \ell \\ 1, & k = \ell \end{cases}$

so the matrix A becomes the Identity, nothing to solve: $c_k = \langle f, \varphi_k \rangle$, $k=1:n$!!!

If we choose <u>orthogonal</u> basis functions then $c_k = \dfrac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}$, $k=1:n$.

The secret is orthogonality !!!

It makes Least Squares approximation easy, just integrations, no system to solve!

The coefficients $c_k = \langle f, \varphi_k \rangle$, $k=1:n$ are called <u>Fourier components of f</u>

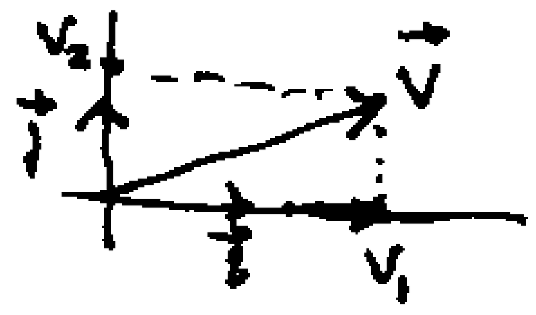
w.r.t. the ON basis $\{\varphi_k\}$, and the representation $f \overset{L^2}{=} \sum_{k=1}^{n} c_k \varphi_k$

is the <u>Fourier expansion of f</u> w.r.t. the ON basis $\{\varphi_k\}_{k=1:n}$

This is the ultimate generalization of expressing a vector by (orthonormal) <u>coordinates</u>.

exactly as $\vec{v} = (v_1, v_2) \equiv v_1 \vec{i} + v_2 \vec{j} \iff v_1 = \langle \vec{v}, \vec{i} \rangle$ = projection of $\vec{v}$ on $\vec{i}$

w.r.t. ON axes $\{\vec{i}, \vec{j}\}$ = coordinate system $\quad v_2 = \langle \vec{v}, \vec{j} \rangle$ = projection of $\vec{v}$ on $\vec{j}$



The mapping $f \to \{c_1, \dots, c_n\}$ of f to its Fourier coefficients is called

the <u>discrete Fourier Transform</u> of f w.r.t. ON basis $\{\varphi_k\}$. encoding

The mapping is invertible: $\{c_k\} \to f$ and gives back $\sum_{k=1}^{n} c_k \varphi_k \overset{L^2}{=} f$!!! decoding (reconstruction)

The encoding $\{c_k\}$ is a compressed representation of the signal f.

Trigonometric Fourier expansion (Fourier series)

For many reasons (historical, mathematical, technological) the standard basis $\{\varphi_k\}$ used in practice is the set of

trigonometric polynomials $\left\{1, \cos\frac{k\pi x}{\ell}, \sin\frac{k\pi x}{\ell}\right\}_{k=1,2,\dots}$ on $(-\ell,\ell)$

which is an orthogonal basis in $L^2(-\ell,\ell)$. The common notation for the

Fourier Series of $f \in L^2(-\ell,\ell)$: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}$

with Fourier coefficients $a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$, $b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$
$n=1,2,\dots$

Complex notation is more convenient: Orthogonal basis: $\left\{e^{i\frac{k\pi x}{\ell}}\right\}_{k=0,\pm1,\pm2,\dots}$

Fourier Series of $f \in L^2(-\ell,\ell)$: $f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi x}{\ell}}$

with Fourier coefficients $\underbrace{c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\, e^{-i\frac{k\pi x}{\ell}}\,dx}_{=\text{amplitude at frequency } k}$, $k=0,\pm1,\dots$

The mapping $f \to \{c_k\}_{k=0,\pm1,\dots}$ is the Discrete Fourier Transform of $f \in L^2(-\ell,\ell)$

Inverse DFT reconstruct f from $\{c_k\}$: $\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi x}{\ell}} \overset{L^2}{=} f(x)$.

The series converges in L^2-sense: $\left\| f - \sum_{k=-m}^{m} c_k e^{i\frac{k\pi x}{\ell}} \right\|_{L^2} \to 0$ as $m\to\infty$
and $\|f\|^2 = \sum_{-\infty}^{\infty} |c_k|^2 < \infty$.

It will also converge pointwise (at each x) if $\{c_k\}$ decay fast enough.

The theory of convergence of Fourier Series is very well developed, in fact it prompted the development of much of mathematical analysis in the 1800s, ever since Fourier introduced Fourier Series in ~1805 to solve the heat conduction equation $\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$.

Theorem: If f is continuous in $[-\ell,\ell]$ and f' is of bounded variation in $(-\ell,\ell)$ and 2ℓ-periodic then the Fourier Series of f converges uniformly to $f(x)$ in $(-\ell,\ell)$.

Fourier Transforms - FFT - signal processing

Fourier Transforms represent a function "in frequency domain", in terms of ~~trigonometric~~ polynomials, (Fourier coefficients)

Inverse Transforms reconstruct the function from its Fourier components.

Various versions are in use:

1. Fourier Integral of $f(x)$ defined in $(-\infty, \infty)$: $F(\nu) = \int_{-\infty}^{\infty} f(x) e^{i 2\pi \nu x} dx$, $-\infty < \nu < \infty$

   Fourier component at frequency ν

   Inverse Fourier Integral: $f(x) = \int_{-\infty}^{\infty} F(\nu) e^{-i 2\pi \nu x} d\nu$

   reconstructs $f(x)$ from its Fourier components

2. Discrete Fourier Transform of $f(x)$ defined in $(-\ell, \ell)$ is a sequence of numbers $\{\hat{f}_k\}_{k=0,\pm 1,\dots}$ $\hat{f}_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{i\frac{k\pi x}{\ell}} dx$, $k = 0, \pm 1, \pm 2, \dots$

   = Fourier component of f at frequency $\nu_k = \frac{k}{2\ell}$

   Note: $\hat{f}_0$ = mean value of $f(x)$ on $(-\ell, \ell)$, DC component of signal f

   Inverse Discrete Fourier Transform: $f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{-i\frac{k\pi x}{\ell}}$, $-\ell < x < \ell$

   = Fourier Series expansion of f

   = 2ℓ-periodic extension of f on $-\infty < x < \infty$

   reconstructs $f(x)$ from its Fourier components $\{\hat{f}_k\}$

3. Finite Fourier Transform of a finite sequence $\{y_j\}_{j=0,1,\dots,N-1}$ is the finite sequence $\{Y_k\}$: $Y_k = \sum_{j=0}^{N-1} y_j e^{i\frac{2\pi jk}{N}}$, $k = 0, 1, \dots, N-1$

   Inverse Finite Fourier Transform: $y_j = \sum_{k=0}^{N-1} Y_k e^{-i\frac{2\pi jk}{N}}$, $j = 0, 1, \dots, N-1$

   reconstructs $\{y_j\}$ from its Fourier components $\{Y_k\}$.

4. Fast Finite Fourier Transform (FFT) implements the computation of Finite FT of length $N = 2^p$ in only $\mathcal{O}(N \cdot p) = \mathcal{O}(N \log_2 N)$ operations instead of $\mathcal{O}(N^2)$ (tremendous savings for large N: e.g. for $N = 2^{10} = 1024$: $N^2 \approx 10^6$, $Np \approx 10^4$) makes FT practical in real time. Considered one of most important algorithms ever!

Digital signal processing in modern technology

1. signal is digitized into $\{y_j\}$ by sampling
2. $\{y_j\}$ is transformed ito frequency domain $\{Y_k\}$ via FFT: $Y = fft(y)$
3. $\{Y_k\}$ manipulated: filtered, denoised, amplified, compressed, etc.
4. transmitted
5. $\{y_j\}$ is reconstructed via Inverse FFT by the receiver!

Amazing technology based on deep mathematis developed over 200 years!