MATRIX FACTORIZATIONS

1 $\mathbf{A = LU} = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix}$ *Section 2.5*

Requirements: No row exchanges as Gaussian elimination reduces A to U.

2 $\mathbf{A = LDU} = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{1's on the diagonal} \end{pmatrix}$

Requirements: No row exchanges. The pivots in D are divided out to leave 1's in U. If A is symmetric then U is the transpose of L and $A = LDL^{\mathrm{T}}$. *Section 2.5*

3 $\mathbf{PA = LU}$ (permutation matrix P to reorder the rows of A).

Requirements: A is invertible. Then P, L, U are invertible. P does the row exchanges in advance. Alternative: $A = L_1 P_1 U_1$. *Section 2.6*

4 $\mathbf{PA = LU} = (m$ by m lower triangular matrix $L)(m$ by n echelon matrix $U)$.

Requirements: None! U has r pivot rows and pivot columns, with zeros below pivots. Complete elimination gives the *reduced echelon form* R. *Section 3.2*

5 $\mathbf{A = CC^T} =$ (lower triangular matrix C) (transpose is upper triangular)

Requirements: A is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* has $C = L\sqrt{D}$. *Section 6.5*

6 $\mathbf{A = QR} =$ (orthonormal columns in Q) (upper triangular R)

Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt process. If A is square then $Q^{-1} = Q^{\mathrm{T}}$. *Section 4.3*

7 $\mathbf{A = S\Lambda S^{-1}} =$ (eigenvectors in S)(eigenvalues in Λ)(left eigenvectors in S^{-1}).

Requirements: A has n linearly independent eigenvectors. *Section 6.2*

8 $\mathbf{A = Q\Lambda Q^T} =$ (orthogonal matrix Q)(real eigenvalue matrix Λ)(Q^{T} is Q^{-1}).

Requirements: A is symmetric. This is the *Spectral Theorem*. *Section 6.4*

9 $\mathbf{A} = \mathbf{MJM}^{-1}$ = (generalized eigenvector matrix M)(Jordan block matrix J)(M^{-1}).

Requirements: A is any square matrix. Number of independent eigenvectors of A is the number of blocks in the *Jordan form* J. Each block has one eigenvalue. *Section 6.6*

10 $\mathbf{A} = \mathbf{U\Lambda U}^{-1}$ = (unitary U)(eigenvalue matrix Λ)(U^{-1} which is $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}}$).

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$. *Section 10.2*

11 $\mathbf{A} = \mathbf{UTU}^{-1}$ = (unitary U)(triangular T with λ's on diagonal)(U^{-1} which is U^{H}).

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular. *Section 10.2*

12 $$\mathbf{A} = \mathbf{U\Sigma V}^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m \times m \end{pmatrix} \begin{pmatrix} m \times n \text{ singular value matrix} \\ \sigma_1, \ldots, \sigma_r \text{ on its diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ V \text{ is } n \times n \end{pmatrix}.$$

Requirements: None. This *singular value decomposition* (SVD) has the eigenvectors of $A^{\mathrm{T}}A$ in U and of AA^{T} in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$. *Section 7.3*

13 $$\mathbf{A}^{+} = \mathbf{V\Sigma}^{+}\mathbf{U}^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ n \times n \end{pmatrix} \begin{pmatrix} n \times m \text{ pseudoinverse of } \Sigma \\ 1/\sigma_1, \ldots, 1/\sigma_r \text{ on diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ m \times m \end{pmatrix}.$$

Requirements: None. The *pseudoinverse* has $A^{+}A$ = projection onto row space of A and AA^{+} = projection onto column space. The shortest least-squares solution to $A\mathbf{x} = \mathbf{b}$ is $\bar{\mathbf{x}} = A^{+}\mathbf{b}$. *Section 7.3*

14 $\mathbf{A} = \mathbf{QH}$ = (orthogonal matrix Q)(symmetric positive definite matrix H).

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD. *Section 7.3*

15 $$\mathbf{F}_n = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{n/2} & \\ & F_{n/2} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}.$$

Requirements: F_n = Fourier matrix with entries w^{jk} where $w^n = 1$. Then $F_n\overline{F}_n = nI$. D has $1, w, w^2, \ldots$ on its diagonal. For $n = 2^l$ the *Fast Fourier Transform* has $\frac{1}{2}nl$ multiplications from l stages of D's. *Section 10.3*

The factorizations 1–8 are in the basic course; 9–15 are optional.