Logic and Proof II

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Logical Equivalences and Set-theoretic Identities

Logical Equivalence	Corresponding Set-theoretic Identity
1. De Morgan's laws	
a. $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	a*. $(A \cap B)^c = A^c \cup B^c$
b. $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	b*. $(A \cup B)^c = A^c \cap B^c$
2. Commutative laws	
a. $(P \wedge Q) \Leftrightarrow (Q \wedge P)$	a*. $A \cap B = B \cap A$
b. $(P \vee Q) \Leftrightarrow (Q \vee P)$	b*. $A \cup B = B \cup A$
c. $(P + Q) \Leftrightarrow (Q + P)$	c*. $A \Delta B = B \Delta A$
3. Associative laws	
a. $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$	a*. $A \cap (B \cap C) = (A \cap B) \cap C$
b. $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$	b*. $A \cup (B \cup C) = (A \cup B) \cup C$
c. $P + (Q + R) \Leftrightarrow (P + Q) + R$	c*. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$
4. Idempotent laws	
a. $P \wedge P \Leftrightarrow P$	a*. $A \cap A = A$
b. $P \vee P \Leftrightarrow P$	b*. $A \cup A = A$
5. Distributive laws	
a. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	a*. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
b. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	b*. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
c. $P \wedge (Q + R) \Leftrightarrow (P \wedge Q) + (P \wedge R)$	c*. $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

6. Absorption laws

a. $P \wedge (P \vee Q) \Leftrightarrow P$ a*. $A \cap (A \cup B) = A$

b. $P \vee (P \wedge Q) \Leftrightarrow P$ b*. $A \cup (A \cap B) = A$

7. Double negation law

$\neg\neg P \Leftrightarrow P$ $(A^c)^c = A$

8. Tautology laws (U = the universal set)

a. $P \vee \neg P$ is a tautology a*. $A \cup A^c = U$

b. $P \wedge$(a tautology) $\Leftrightarrow P$ b*. $A \cap U = A$

c. $P \vee$ (a tautology) is a tautology c*. $A \cup U = U$

d. P + (a tautology) $\Leftrightarrow \neg P$ d*. $A \,\Delta\, U = A^c$

e. $\neg$(a tautology) is a contradiction e*. $U^c = \varnothing$

9. Contradiction laws (U = the universal set)

a. $P \wedge \neg P$ is a contradiction a*. $A \cap A^c = \varnothing$

b. $P \wedge$(a contradiction) is a contradiction b*. $A \cap \varnothing = \varnothing$

c. $P \vee$ (a contradiction) $\Leftrightarrow P$ c*. $A \cup \varnothing = A$

d. $\neg P \to$ (a contradiction) $\Leftrightarrow P$ This equivalence, which justifies proof by contradiction, is just a variant of the preceding equivalence, since $\neg P \to Q \Leftrightarrow P \vee Q$.

e. P + (a contradiction) $\Leftrightarrow P$ d*. $A \,\Delta\, \varnothing = A$

f. $\neg$(a contradiction) is a tautology e*. $\varnothing^c = U$

10. Conditional equivalences

    a. $P \to Q \Leftrightarrow \neg P \vee Q$

    b. $\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$

    c. $P \to (Q \to R) \Leftrightarrow (P \wedge Q) \to R$

11. Biconditional equivalences

    a. $P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

    b. $\neg(P \leftrightarrow Q) \Leftrightarrow P + Q$

12. Quantifier laws

    a. $\neg\forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x)$

    a*. $(\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c$

    b. $\neg\exists x\, P(x) \Leftrightarrow \forall x \neg P(x)$

    b*. $(\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c$

    c. $\forall x\, (P(x) \wedge Q(x)) \Leftrightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$

    d. $\forall x\, P(x) \vee \forall x\, Q(x) \Rightarrow \forall x\, (P(x) \vee Q(x))$

    e. $\forall x\, (P(x) \vee Q(x))$ DOES NOT IMPLY $\forall x\, P(x) \vee \forall x\, Q(x)$!!!

    f. $\exists x\, (P(x) \vee Q(x)) \Leftrightarrow \exists x\, P(x) \vee \exists x\, Q(x)$

    g. $\exists x\, (P(x) \wedge Q(x)) \Rightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$

    h. $\exists x\, P(x) \wedge \exists x\, Q(x)$ DOES NOT IMPLY $\exists x\, (P(x) \wedge Q(x))$!!!