MATH 231, FALL 2022/Section 7/ FIRST MIDTERM/ October 4, 2022
(Closed book, no notes, no calculators. 5 pts per item. Time given: 75 min.)

1. Find the general solution $y = y(x), x > 0$:

$$(i)\frac{dy}{dx} = \frac{1}{y} \quad (ii)\frac{dy}{dx} = \sqrt{x+y} - 1, \quad (iii)\frac{dy}{dx} + \frac{y}{x} = y^{1/2}.$$

2. Solve the initial-value problems (in implicit form, if necessary.)

(i) $(\frac{1}{x} + 2y^2x)dx + (2yx^2 - \cos y)dy = 0, \quad y = y(x), x > 0, y(1) = \pi.$

(ii) $y' + \frac{3y}{x} + 4 = 5x, y = y(x), y(1) = 1$ (include the domain of definition of the solution)

3. For the autonomous first-order equation below:

$$y' = y^2(y+2)(y-2), \quad y = y(t).$$

(i) Find and classify the equilibria (=constant solutions) as stable, unstable or semi-stable, plotting them on the phase line (with arrows indicating the direction of motion.)

(ii) Sketch the graph of one solution in each of the ranges defined by the equilibria, indicating those that "blow up" at some finite positive or negative time (that is, have a vertical asymptote).

4. (i) The half-life of the radioactive carbon isotope C^{14} is 5,700 years. An animal bone is found to contain a fraction of C^{14} (relative to the stable isotope C^{12}) about 12.5% (or 1/8) of the current atmospheric ratio of C^{14} to C^{12}, assumed to be about the same as when the animal died. Estimate how long ago that happened.

(ii) The equation of motion for the velocity $v(r)$ of a body thrown upwards from the surface of a planet (as a function of the distance r from the body to the center of the planet) is: $v\frac{dv}{dr} = -\frac{10R^2}{r^2}$, where $R = 10^3 km$ is the radius of the planet. Solve this equation of motion (for a given $v(R)$) to find the smallest value of $v(R)$ that guarantees $v(r) > 0$ for all $r \geq R$ (that is, the body does not return.)

5. Both functions $y_1(x), y_2(x)$ below are solutions of the initial-value problem (IVP): $y'(x) = 3y^{2/3}, \quad y = y(x), \quad y(0) = 0.$

$$y_1(x) = x^3, x \in \mathbb{R}, \qquad y_2(x) = x^3, x \geq 0; \quad y_2(x) = 0, x \leq 0.$$

Explain why this does not conflict with the conclusion of the existence-uniqueness theorem for first-order IVP $y' = f(x, y), \quad y = y(x), \quad y(x_0) = y_0$.