1

(i) $\frac{dy}{dx} = \frac{1}{y}$ $\quad y\,dy = dx$ (sep'ble) $\quad \int y\,dy = \frac{y^2}{2} = \int dx = x + c$

$\boxed{y^2 = 2x + C}$, $C \in \mathbb{R}$, $x > \frac{-C}{2}$

(ii) $\frac{dy}{dx} = \sqrt{x+y} - 1$ $\quad$ set $v = x + y$ $\quad v' = 1 + y'$

$v' - 1 = \sqrt{v} - 1$ $\quad \frac{dv}{\sqrt{v}} = dx$ (sep'ble) $\quad 2\sqrt{v} = x + C$

$\sqrt{v} = \frac{x}{2} + C$, $\quad v = \left(\frac{x}{2} + C\right)^2$

$\boxed{y = \left(\frac{x}{2} + C\right)^2 - x \quad x \in \mathbb{R} \quad C \in \mathbb{R}}$

(iii) $y' + \frac{y}{x} = y^{1/2}$ $\quad$ Bernoulli, $n = 1/2$, set $v = y^{1-n} = y^{1/2}$

$v' = \frac{1}{2} y^{-1/2} y'$

$\frac{1}{2} y^{-1/2} y' + \frac{1}{2} \frac{y^{1/2}}{x} = \frac{1}{2}$

$v' + \frac{1}{2}\frac{v}{x} = \frac{1}{2}$ (linear: int factor $\mu = e^{\int \frac{dx}{2x}} = x^{1/2}$)

$(x^{1/2} v)' = x^{1/2} v' + \frac{1}{2} x^{-1/2} v = \frac{x^{1/2}}{2} \rightarrow x^{1/2} v = \int \frac{x^{1/2}}{2} dx = \frac{1}{3} x^{3/2} + C$

$v = \frac{1}{3} x + C x^{-1/2}$ $\quad y = v^2 = \left(\frac{x}{3} + \frac{C}{\sqrt{x}}\right)^2$ // gen'l sln $x > 0$, $C \in \mathbb{R}$.

2

(i) $\left(\frac{1}{x} + 2y^2 x\right) dx + (2yx^2 - \cos y)\, dy = 0$ $\quad y(1) = \pi$

$M_y = 4xy = N_x \Rightarrow$ exact

$F(x,y) = \int (2yx^2 - \cos y)\, dy = x^2y^2 \mp \sin y + C(x)$

$F_x = 2xy^2 + \frac{dC}{dx} = 2xy^2 + \frac{1}{x} \rightarrow C(x) = \ln x$

$F(x,y) = x^2y^2 + \sin y + \ln x = \text{const}$

$F(1,\pi) = \pi^2 + \sin\pi + \ln 1 = \pi^2 \Longrightarrow \text{const} = \pi^2$

$\boxed{x^2y^2 \mp \sin y + \ln x = \pi^2}$, $x > 0$

(ii) $y' + 3\frac{y}{x} + 4 = 5x$ (linear) $\quad \mu(x) = e^{\int \frac{3}{x} dx} = x^3$

$(x^3 y)' = x^3 y' + 3x^2 y = 5x^4 - 4x^3$

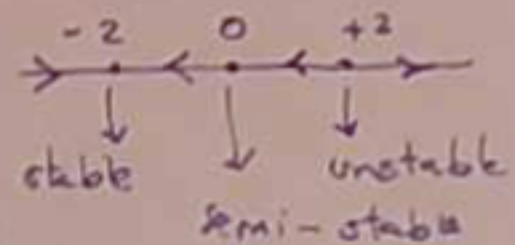
$x^3 y = \int (5x^4 - 4x^3)\, dx = x^5 - x^4 + C$

$y(1) = 1 \longrightarrow C = 1$

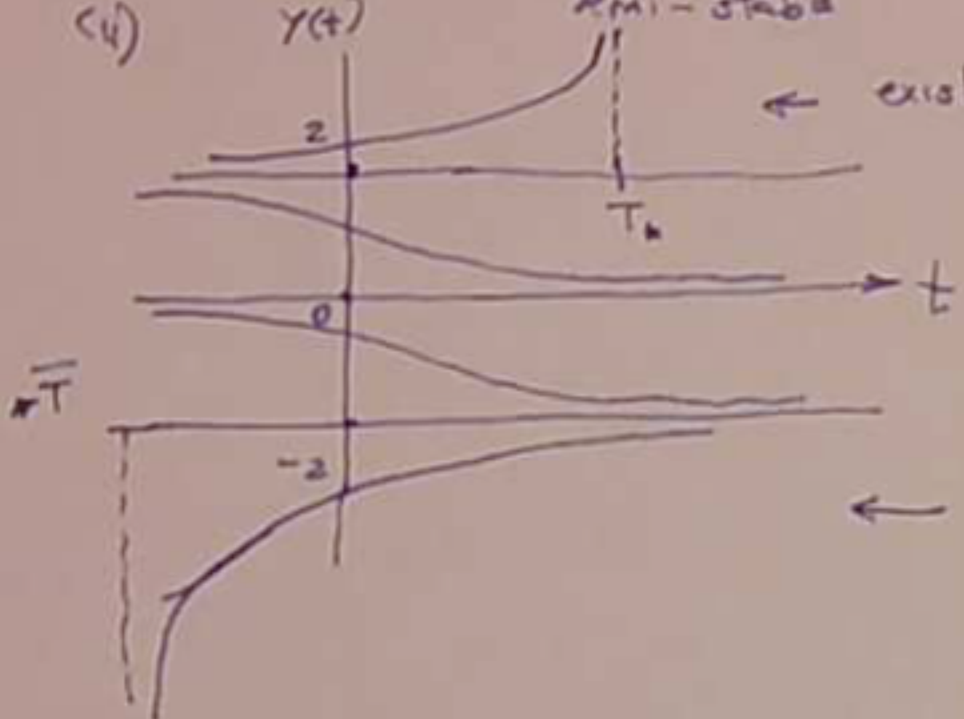
$\boxed{y(x) = x^2 - x + \frac{1}{x^3}, \quad x > 0}$

3. $y' = f(y) = y^2 (y+2)(y-2)$: eq. at $-2, 0, +2$

(i)



(ii)



← existence on $(-\infty, T_*)$

← existence on $(\overline{T}, +\infty)$

4. (i) $\frac{1}{8} = \left(\frac{1}{2}\right)^3$, 3 half-lives : $3 \times 5{,}700 = 17{,}100$ yrs.

(ii) $v \frac{dv}{dr} = -\frac{10R^2}{r^2}$, $v\,dv = -(10R^2)\frac{dr}{r^2}$ (cf. p. 66)

$$\int v\,dv = \frac{1}{2}v^2 = -(10R^2)\int \frac{dr}{r^2} = \frac{10R^2}{r} + C$$

$$v^2(r) = \frac{20R^2}{r} + C \qquad v^2(R) = 20R + C \rightarrow -C = 20R - v^2(R)$$

$$v^2(r) = \frac{20R}{r} + v^2(R) - 20R \rightarrow v(r) > 0 \text{ if } v^2(R) > 20R$$

or $v(R) > \sqrt{20R}$

5. Uniqueness requires $\frac{\partial f}{\partial y}(x_0, y_0)$ defined and cont.

But $\frac{\partial f}{\partial y} = 2y^{-1/3}$ not defined at $y_0 = 0$, so the thm. does not allow us to conclude uniqueness of slns.