MATH 231 – 9/1/2022

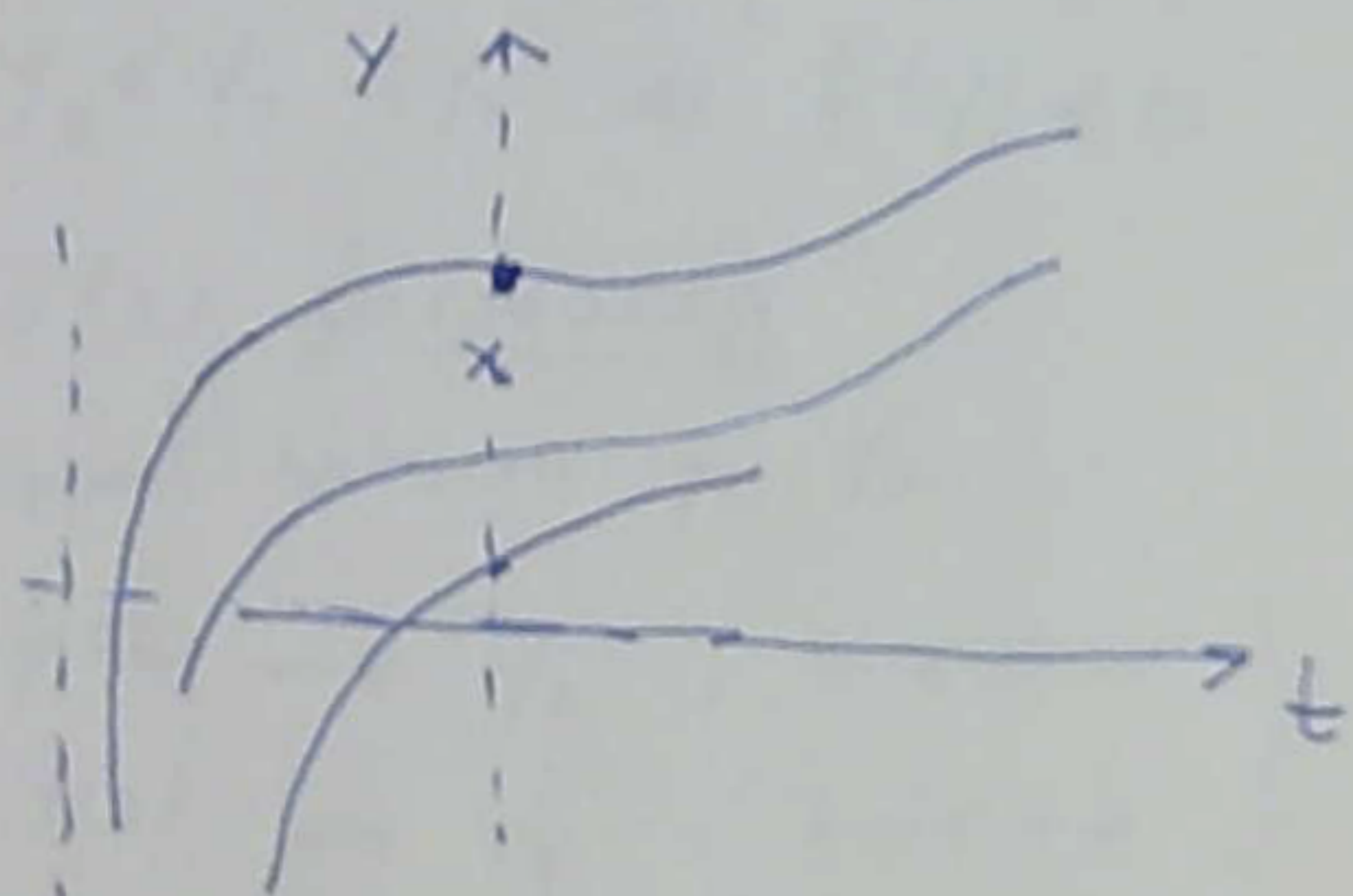
Phase line

graphical method to understand 1st order autonomous D.E.

I.V.P. $\begin{cases} y' = f(y), & y = y(t) \\ y(0) = y_0 \end{cases}$ f cont., f' cont on $y \in \mathbb{R}$

physically given initial position, & law of motion (velocity = f(position))
find $y(t)$ (position at time t)

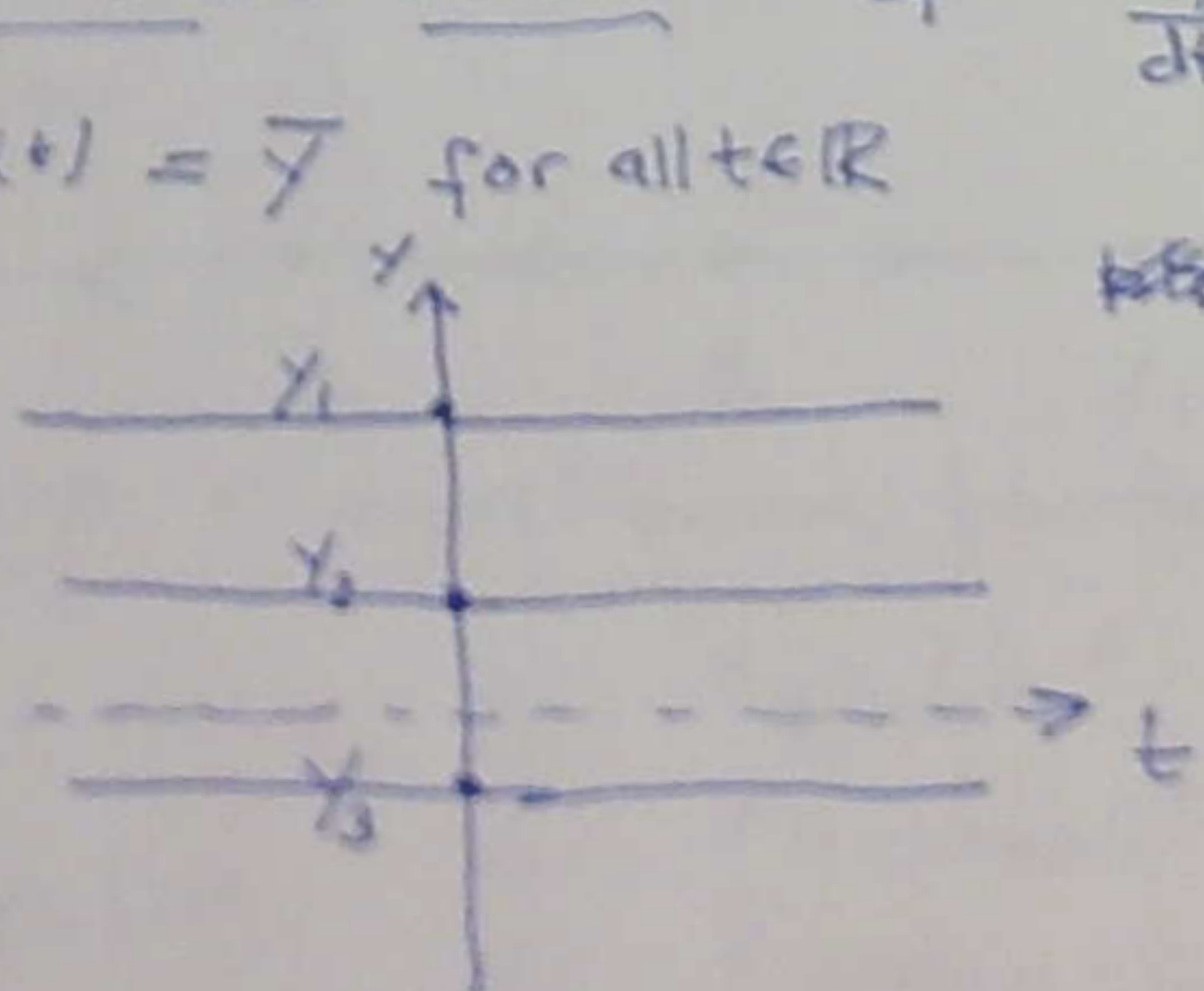
geometric interp. of E–U theorem ($\forall\, y_0 \in \mathbb{R}$, $\exists$ a sln $y(t)$ defined on some interval $(-\delta, \delta)$, and only one)



- there is a unique solution curve through each $(0, y_0)$
- two solution curves ~~may not~~ never intersect

I.V.P.
$y(t_0) = y_0$
would have two slns

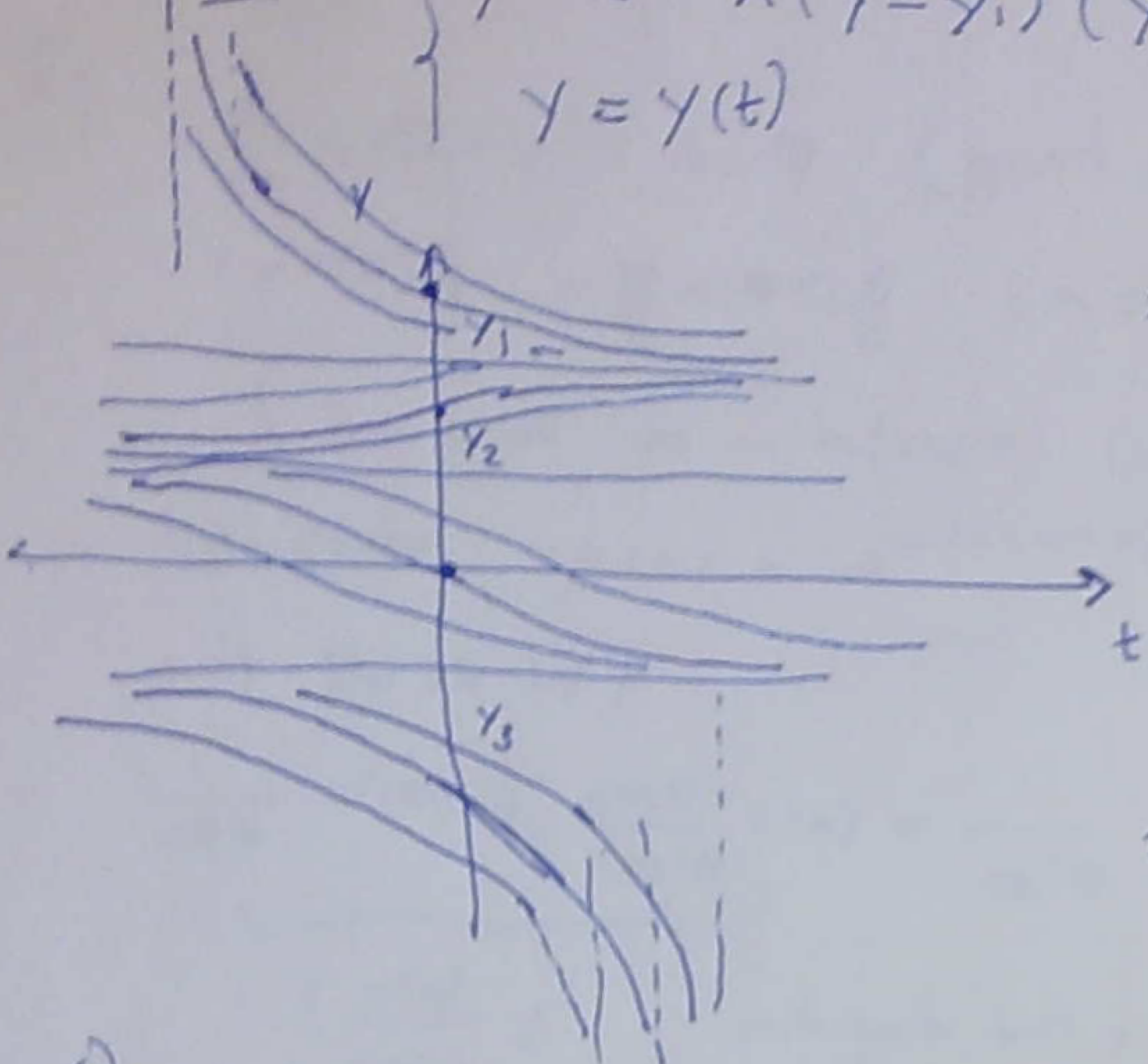
Any $\bar{y}$ s.t. $f(\bar{y}) = 0$ is
a constant solution of $\frac{dy}{dt} = f(y)$
$y(t) = \bar{y}$ for all $t \in \mathbb{R}$



① plot zeroes of f on y-axis
↓
const. slns (horiz. lines)

② analyze the sign of f between zeroes

Ex. $\left\{\begin{array}{l} y' = \overbrace{-A(y-y_1)(y-y_2)(y-y_3)^2}^{f(y)} \\ y = y(t) \end{array}\right.$ $(A>0)$



1) const. slns y_1, y_2, y_3

2) $y > y_1 \Rightarrow f(y) < 0$
$\Rightarrow y' < 0$
$\Rightarrow y(t)$ <u>decreasing</u>

$y(t) \to y_1$ (from above as $t \to +\infty$)

blows up as $t \to -t_1$ where t_1 depends on $y(0)$

3) $y_2 < y_0 < y_1 \Rightarrow f(y) > 0$

$y(t)$ <u>increasing</u>

"trapped between const. slns y_1, y_2"

$y(t) \longrightarrow y_1 \quad t \to +\infty$

$y(t) \to y_2 \quad t \to -\infty$

$y(t)$ <u>defined</u> for all $t \in \mathbb{R}$

4) $y_3 < y_0 < y_2 \Rightarrow f(y) < 0$

$y(t)$ <u>decreasing</u>

trapped between y_2, y_3

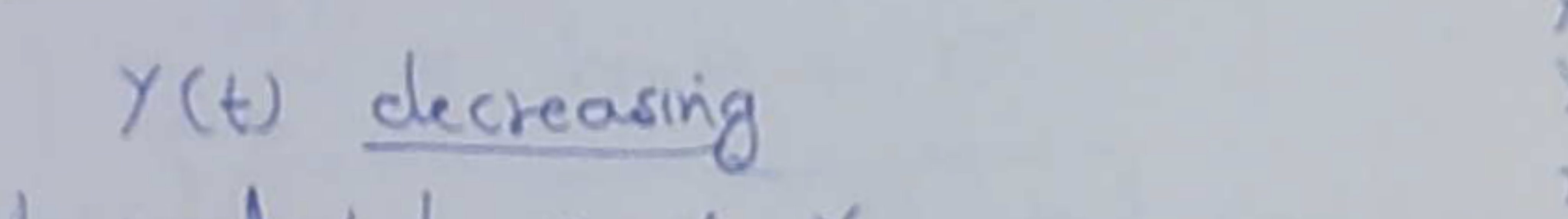
$\left.\begin{array}{l} y(t) \longrightarrow y_3, \quad t \to +\infty \\ y(t) \to y_2, \quad t \to -\infty \end{array}\right\}$ $y(t)$ defined for all $t \in \mathbb{R}$

5) $y_0 < y_3 \longrightarrow f(y) < 0 \longrightarrow y(t)$ <u>decreasing</u>

$y(t) \to y_3 \quad t \to -\infty$

vertical asymptotes (blowup in finite time) for t increasing

<u>phase line</u>



"motion of a particle under the law $y' = f(y)$"

<u>const. slns</u> "equilibrium points" "equilibria"

y_1 : "stable" equilibrm.

<u>phase line</u> y_2 : "unstable" eq.

y_3 : "semi-stable" eq.

2.3, p. 54

9

$$\frac{dr}{d\theta} + r\tan\theta = \sec\theta \quad (\text{gen'l sln.})$$

$r = r(\theta)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ (in std. form, $p(\theta) = \tan\theta$)

$$\int \tan\theta \, d\theta = -\ln(\cos\theta) \quad (+C)$$

int. factor $\mu(\theta) = e^{-\ln(\cos\theta)} = e^{\ln(1/\cos\theta)} = \frac{1}{\cos\theta}$

mult. the DE by μ:

$$\underbrace{\frac{1}{\cos\theta} r'(\theta) + \frac{\sin\theta}{\cos^2\theta} r(\theta)}_{\left(\frac{r(\theta)}{\cos\theta}\right)'} = \frac{1}{\cos^2\theta}$$

integrate both sides $\int \sec^2\theta \, d\theta = \tan\theta + C$ (TABLE)

$$\frac{r(\theta)}{\cos\theta} = \tan\theta + C$$

$$\boxed{r(\theta) = \sin\theta + C\cos\theta} \quad (\text{gen'l sln}: C \in \mathbb{R})$$

check

$C = 0$ $r(\theta) = \sin\theta$

$$r'(\theta) + r\tan\theta = \cos\theta + \frac{\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta} \checkmark$$

for $r(\theta) = \cos\theta$

$$r'(\theta) + r\tan\theta = -\sin\theta + \cos\theta \cdot \tan\theta = 0 \checkmark \quad (\text{assoc. hom. eqn})$$

20

$$\frac{dy}{dx} + \frac{3y}{x} + 2 = 3x, \quad y(1) = 1 \quad (\text{sln. of I.V.P.})$$

$$y' + \frac{3}{x}y = 3x - 2 \quad (\text{std. form}) \quad y = y(x), \quad x > 0$$

$p(x) = \frac{3}{x}$, $\int p(x)\,dx = \int \frac{3}{x}\,dx = 3\ln x = \ln x^3$

int. factor $\mu(x) = e^{\ln x^3} = x^3$

$(\times x^3)$

$$\underbrace{x^3 y' + 3x^2 y}_{(x^3 y)'} = 3x^4 - 2x^3 \xRightarrow{\int} x^3 y = \frac{3}{5}x^5 - \frac{1}{2}x^4 + C$$

at $x=1, y=1$ ($y(1)=1$ given)

$$1 = \frac{3}{5} - \frac{1}{2} + C = \frac{1}{10} + C \to C = 9/10$$

$$y(x) = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{9}{10}\frac{1}{x^3} \quad (x > 0) \quad //$$

Sect 1.2

~~9~~ $\boxed{10}$

$$y - \ln y = x^2 + 1 \quad , \quad \frac{dy}{dx} = \frac{2xy}{y-1}$$

implicit sln ? (check)

$$G(x,y) = y - \ln y - x^2 = 1 \quad \text{(a constant)}$$

to check if $y = y(x)$ is a sln of the D.E, does it follow that $G(x, y(x)) = \text{const.}$

(i.e. is $\frac{d}{dx} G(x, y(x)) = 0$?)

$$\frac{d}{dx}\left[G(x, y(x))\right] = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx}$$

" implicit differentiation in x ".

$$= -2x + \left(1 - \frac{1}{y}\right) y'$$

$$= -2x + \frac{y-1}{y} \cdot \frac{2xy}{y-1} = 0$$

so $G(x,y)=1$ is an implicit sln. of this D.E.

$\boxed{12}$ $x^2 - \sin(x+y) = 1$ D.E. $y' = 2x \sec(x+y) - 1$

$G(x,y)$.

$$\frac{d}{dx} G(x, y(x)) = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot y' = 2x - \cos(x+y) - \cos(x+y)\, y'$$

$$= 2x - \cos(x+y) - \left(\frac{2x}{\cos(x+y)} - 1\right) \cos(x+y) = 0$$

so the $G(x,y) = 1$ is an implicit sln. of the given D.E.

$\boxed{24}$ $\begin{cases} y'(t) - ty = \sin^2 t \\ y = y(t), \quad y(0) = 7 \end{cases}$ Apply E-U thm, state conclusion for this I.V.P.

$y'(t) = ty + \sin^2 t$

$f(t, y) = ty + \sin^2 t$

$\frac{\partial f}{\partial y} = t$ cont. for all (t, y).

The I.V.P. w/ I.C. $y(0) = 7$ has a unique sln, defined on some interval $t \in (-\delta, \delta)$, $\delta > 0$

note in fact the sln is defined for all $t \in \mathbb{R}$ in this case.
since this a $\boxed{\text{linear}}$ 1st order eqn (yields $\mu(t) = e^{+t^2/2}$)

$\boxed{28}$ $y'(x) = \underbrace{3x - \sqrt[3]{y-1}}_{f(x,y)}$ $\quad y(2) = 1$

$\frac{\partial f}{\partial y} = -\frac{d}{dy}(y-1)^{1/3} = -\frac{1}{3}(y-1)^{-2/3}$ $\boxed{\text{if}}$ $y \neq 1$.
(not defined if $y = 1$)

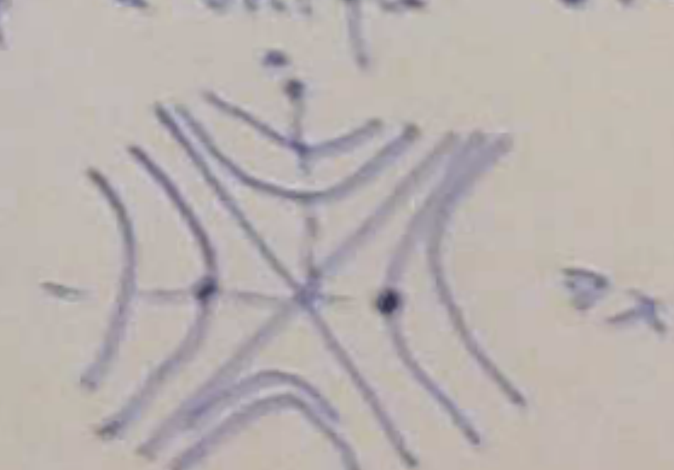
the E-U thm does not guarantee the existence of a unique sln to this I.V.P.
(there exists more than one sln.)

$\boxed{31}$ $yy' - 4x = 0$, $y(x_0) = 0$

$y' = \frac{4x}{y} = f(x,y)$ $\quad \frac{\partial f}{\partial y} = -\frac{4x}{y^2}$, $y \neq 0$ (not defined if $y = 0$)

$\rightarrow$ cannot conclude the existence of a unique sln.

implicit slns $4x^2 - y^2 = \text{const.}$



curves through $(x_0, 0)$ are not graphs $y = y(x)$