

Green's functions, oscillation, Sturm-Liouville eigenvalues.

We consider the Dirichlet BVP for the general second-order operator:

$$L[y] = y'' + b(t)y' + q(t)y, \quad b, q \in C(I), y = y(t), t \in I$$

or the self-adjoint operator:

$$L[y] = (p(t)y')' + q(t)y, \quad p, p', q \in C(I), p(t) > 0.$$

1. *Disconjugacy.* The closed interval $[a, b] \subset I$ is *disconjugate* for the operator L if no nontrivial solution to $Ly = 0$ vanishes at two points $t_1, t_2 \in [a, b]$.

2. *The homogeneous Dirichlet BVP.* The problem $L[y] = 0, y(a) = A, y(b) = B$ has a unique solution for all A, B if and only if $[a, b]$ is disconjugate for L .

Proof. Let $y_1(t), y_2(t)$ be l.i. solutions of $L[y] = 0$ in I . Then existence/uniqueness for the system:

$$c_1y_1(a) + c_2y_2(a) = A, \quad c_1y_1(b) + c_2y_2(b) = B$$

for arbitrary A, B is equivalent to:

$$y_1(a)y_2(b) - y_1(b)y_2(a) \neq 0.$$

This condition is equivalent to saying the problem with $A = B = 0$ has only the trivial solution $c_1 = c_2 = 0$.

3. *The Dirichlet Green's function for $[a, b]$.* Let $[a, b] \subset I$ be a disconjugate interval and φ, ψ be nontrivial solutions of $L[y] = 0$ satisfying: $\varphi(a) = 0, \psi(b) = 0$. For a given parameter $s \in [a, b]$, define the function of $t \in [a, b]$:

$$G(t, s) = \frac{\varphi(t)\psi(s)}{p(s)W(s)}, a \leq t \leq s; \quad G(t, s) = \frac{\psi(t)\varphi(s)}{p(s)W(s)}, s \leq t \leq b.$$

Here $W(s) = \varphi(s)\psi'(s) - \psi(s)\varphi'(s)$ is the Wronskian of φ, ψ and the denominator $p(s)W(s)$ is *constant* in s . Note that it is nonzero in $[a, b]$, otherwise $[a, b]$ wouldn't be disconjugate. Also, although φ, ψ are only defined up to a constant, $G(t, s)$ is independent of these choices, and depends only on L (and on the interval.)

Properties. $G(t, s)$ is continuous in the square $[a, b] \times [a, b]$, with $G(a, s) = G(b, s) = 0 \forall s$.

The derivative $\partial_t G(t, s)$ has a jump $\partial_t G(s_+, s) - \partial_t G(s_-, s) = 1/p(s)$ at $t = s$.

For any $s \in [a, b]$, $G \in C^2$ in the intervals (a, s) and (s, b) , with $L[G] = 0$ there (as a function of t).

$G(t, s) < 0$ for $t \in (a, b)$.

4. *Solution of the non-homogeneous problem.* If $[a, b] \subset I$ is disconjugate, the non-homogeneous problem (for a given $e(t) \in C([a, b])$):

$$L[y] = e(t), \quad y(a) = 0, y(b) = 0,$$

has a unique solution, given by:

$$y(t) = \int_a^b G(t, s)e(s)ds.$$

(Conversely, if the problem has a unique solution for any $e(t)$, then $[a, b]$ is disconjugate.)

Proof. Split the integral into two parts:

$$y(t) = \psi(t) \int_a^t \frac{\varphi(s)}{p(s)W(s)} e(s)ds + \varphi(t) \int_t^b \frac{\psi(s)}{p(s)W(s)} e(s)ds$$

and compute $L[y]$:

$$py' = p(t)\psi'(t) \int_a^t \frac{\varphi e}{pW} ds + p\psi \frac{\varphi e}{pW}(t)e + p(t)\varphi'(t) \int_t^b \frac{\psi e}{pW} ds - p\varphi \frac{\psi e}{pW}(t);$$

$$(py')' = (p\psi)' \int_a^t \frac{\varphi e}{pW} ds + (p\varphi)' \int_t^b \frac{\psi e}{pW} ds + \frac{\psi'\varphi - \varphi'\psi}{W}(t)e(t).$$

$$L[y] = L[\psi] \int_a^t \frac{\varphi e}{pW} ds + L[\varphi] \int_t^b \frac{\psi e}{pW} ds + e(t) = e(t).$$

5. *Examples.*

a) $L[y] = y'' + y$ in $[0, \frac{\pi}{2}]$.

$$G(t, s) = -\sin t \cos s, t \in [0, s]; \quad G(t, s) = -\cos t \sin s, t \in [s, \frac{\pi}{2}].$$

b) $L[y] = y'' - y$ in $[0, 1]$.

$$G(t, s) = \frac{1}{\sinh 1} \sinh t \sinh(s-1), t \leq s; \quad \frac{1}{\sinh 1} \sinh s \sinh(t-1), s \geq t.$$

c) $L[y] = (py')'$ in $[a, b]$ (where $p(t) > 0$ in $[a, b]$).

$$G(t, s) = -\frac{\int_a^t p^{-1} \int_s^b p^{-1}}{\int_a^b p^{-1}}, t \leq s; \quad G(t, s) = -\frac{\int_t^b p^{-1} \int_a^s p^{-1}}{\int_a^b p^{-1}}, t \geq s.$$

d) $L[y] = y''$ (special case of c) when $p \equiv 1$:

$$G(t, s) = -\frac{(t-a)(b-s)}{b-a}, t \leq s; \quad G(t, s) = -\frac{(b-t)(s-a)}{b-a}, t \geq s.$$

Claim: Note we have $G(t, s) \geq -(b-a)/4$, with equality only at $t = s = (a+b)/2$.

6. *Disconjugacy criteria.* a) If $L[y] = y'' + q(t)y$ and $q(t) \leq 0$ in $[a, b]$, $[a, b]$ is disconjugate. This follows from Sturm's comparison theorem (to be discussed.)

b) *Liapunov's criterion.* For the same operator L , if $q(t)$ satisfies:

$$\int_a^b q_+(t)dt \leq \frac{4}{b-a},$$

then $[a, b]$ is disconjugate. Here $q_+(t) = \max\{q(t), 0\}$.

Proof. Suppose we have $y'' + q(t)y = 0$, with $y(t_1) = y(t_2) = 0$, $q \leq t_1 < t_2 \leq b$. Then $y'' = -q(t)y(t)$, hence:

$$y(t) = -\int_{t_1}^{t_2} G(t, s)q(s)y(s)ds,$$

where $G(t, s)$ is Green's function for the interval $[t_1, t_2]$. We may assume $y(t) > 0$ in (t_1, t_2) , and attains its maximum at $t_0 \in (t_1, t_2)$. Then:

$$y(t_0) = \int_{t_1}^{t_2} |G(t_0, s)|q(s)y(s)ds \leq \int_{t_1}^{t_2} |G(t_0, s)|q_+(s)y(s)ds < \int_{t_1}^{t_2} y(t_0)|G(t_0, s)|q_+(s)ds,$$

where the inequality is strict since we can't have $y(t) \equiv y(t_0)$ in $[t_1, t_2]$. Since $y(t_0) > 0$, it follows from $|G(t, s)| \leq (t_2 - t_1)/4 \leq (b-a)/4$ that:

$$1 < \int_{t_1}^{t_2} |G(t_0, s)|q_+(s)y(s)ds \leq \frac{b-a}{4} \int_a^b q_+(s)ds,$$

contradicting the hypothesis.

7. *Examples.* (i) $x'' + tx = 0$ is disconjugate in $[0, T]$, if $T = 2$.

(ii) $x'' + k \sin(t)x$ is disconjugate on $[a, a + 2\pi]$, if $0 < k < 1/\pi$.

(Note $\int_a^{a+2\pi} q_+(t)dt = k \int_0^\pi \sin t dt = 2k$.)

8. *Sturm separation theorem.* Let u, v be linearly independent solutions of $y'' + a(t)y' + b(t)y = 0$. Then the zeros of u, v alternate (i.e., between two consecutive zeros $x_1 < x_2$ of u , there is a zero of v).

Proof. u' has different signs at x_1, x_2 , and $W(x_i) = (uv' - u'v)(x_i) = -u'v(x_i)$ has the same sign, so $v(x_i)$ has different signs, and v must have a zero between x_1 and x_2 .

Application. If there exists a solution of $L[y] = 0$ which is positive on $[a, b]$, then this interval is disconjugate for L . And conversely, if it is disconjugate, there is a positive solution (take solutions u, v with $u(a) = v(b) = 0, u'(a) = 1, v'(b) = -1$. Then $u > 0$ on (a, b) and $v > 0$ on $[a, b)$, so $u + v > 0$ on $[a, b]$).

9. *Sturm comparison theorem.* Let u, v be (real) solutions of:

$$L_1[u] = (pu')' + q_1(t)u = 0, \quad L_2[v] = (pv')' + q_2(t)v = 0, \quad (p(t) > 0).$$

Assume $q_1 \geq q_2$ on I . Then “ u has more zeros than v ”, in the following sense: between two consecutive zeros $x_1 < x_2$ of v , there is at least one zero of u . (Unless $q_1 \equiv q_2$ and $u = (\text{const})v$).

Proof. Suppose u has no zeros in (x_1, x_2) ; we may assume $u(x_1) \geq 0, u(x_2) \geq 0$, and also $v'(x_1) > 0, v'(x_2) < 0$. Taking the difference, we find:

$$(pu'v - pv'u)' + (q_1 - q_2)uv = 0,$$

and integrating:

$$\int_{x_1}^{x_2} (q_2(t) - q_1(t))dt = (pu'v - pv'u)|_{x_1}^{x_2} = -pv'u|_{x_1}^{x_2} \geq 0.$$

This is impossible, unless $q_1 \equiv q_2$ in $[x_1, x_2]$, and then also (by the separation theorem) $u \equiv (\text{const.})v$ in this interval.

Applications. 1. If $q(t) \leq 0$ in $[a, b]$, any solution $y(t)$ of $(py')' + q(t)y$ has at most one zero in $[a, b]$; in particular, any closed sub-interval of $[a, b]$ is disconjugate.

Proof. Compare with the equation $(p(t)z')' = 0$. The solution $z(t) = \int_a^x \frac{1}{p}(t)dt$ vanishes only at a .

2. A solution of $L[y] = 0$ is *oscillatory* on an interval $J \subset I$ if it has infinitely many zeros in J . (I is the open interval where the coefficients p, q

are continuous). In this case J cannot be a compact interval $[a, b]$ (see [KP, Thm 5.43] for the short proof.) Also, if one solution is oscillatory in J , then all are (by the separation theorem.) So we also say ‘ L is oscillatory in J .’

From the comparison theorem, it follows that if $q_1(t) \geq q_2(t)$ and L_2 is oscillatory in J , then so is L_1 . If an interval $[a, b]$ is disconjugate for L_1 , then it is also disconjugate for L_2 (use the fact that $L_2[y] = 0$ has a solution that is positive in $[a, b]$.)

3. There is a useful extension (found in [KP, Thm 5.50]), allowing the coefficient $p(t)$ to vary. Let

$$L_1[y] = (p_1(t)y')' + q_1(t)y = 0, L_2(z) = (p_2(t)z')' + q_2(t)z = 0.$$

Then, if: $0 < p_2(t) \leq p_1(t)$, $q_1(t) \geq q_2(t)$ in I , between two consecutive zeros of z , there is a zero of y . (Unless $p_1 \equiv p_2$, $q_1 \equiv q_2$ in I and $y = (\text{const.})z$.)

4. The most common comparison is with the constant-potential operator $L[y] = y'' + \omega^2 y$, where the solutions are trigonometric (hence oscillatory in R .) For example, if $q(t) \geq \omega^2 > 0$ on $[a, \infty)$, then L is oscillatory on $[a, \infty)$. Or suppose we have two-sided bounds:

$$c^2 \leq q(t) \leq K^2 \text{ for } t \in I, (\text{with } p(t) \equiv 1).$$

Then by comparison we conclude the distance between consecutive zeros $t_1 < t_2$ of a solution of $L[y] = 0$ satisfies:

$$\frac{\pi}{K} \leq t_2 - t_1 \leq \frac{\pi}{c}.$$

If $y(a) = y(b) = 0$ and y has $n - 1$ zeros in the open interval (a, b) , then:

$$\frac{c(b-a)}{\pi} \leq n \leq \frac{K(b-a)}{\pi}.$$

5. Another useful oscillatory equation for applications of Sturm comparison corresponds to a Cauchy-Euler type equation:

$$L[y] = (t^2 y')' + by = 0 \Rightarrow t^2 y'' + 2ty' + by = 0, t > 0.$$

The characteristic equation is: $r(r-1) + 2r + b = r^2 + r + b = 0$, and if $b > 1/4$ it has complex roots $\alpha \pm i\beta$, leading to the general solution:

$$y(t) = t^\alpha (c_1 \cos(\beta \log t) + c_2 \sin(\beta \log t)).$$

This is clearly oscillatory in $(0, \infty)$, with the distance between consecutive zeros increasing exponentially fast. So the same is true if $0 < p(t) \leq t^2$ and $q(t) \geq b > 1/4$ in $[1, \infty)$ [KP, Ex. 5.52].

10. *A useful change of variable.* Let $s(t) = \int_a^t \frac{d\tau}{p(\tau)}$. This defines an invertible function from $[a, b]$ to $[0, L]$, where $L = \int_a^b \frac{d\tau}{p(\tau)}$. Then if $y(t)$ solves

$$(py')' + q(t)y = 0 \text{ in } [a, b]$$

and we define $x(s) = y(t(s))$, $s \in [0, L]$, we find that $x(s)$ is a solution of:

$$\ddot{x}(s) + Q(s)x(s) = 0, x \in [a, b],$$

where $Q(s) = p(t(s))q(t(s))$. (Here dots denote derivatives with respect to s .)

11. *Fite-Wintner oscillation theorem.* Let $L[y] = (py')' + q(t)y$, $p(t) > 0$, $q(t) > 0$ in $[a, \infty)$. If we have:

$$\int_a^\infty \frac{1}{p(t)} dt = \infty, \quad \int_a^\infty q(t) dt = \infty,$$

then any solution of $L[y] = 0$ is oscillatory in $[a, \infty)$.

Proof. (i) Consider first the case $p(t) \equiv 1$. Arguing by contradiction, suppose the solution with $y(a) = 0, y'(a) = 1$ has no zeros on $\{t > a\}$. We claim there exists a $b > a$ with $y'(b) < 0$. If so, there must be a zero of y'' strictly between a and b (where y' has a local max), and from the equation $y'' = -qy$ (where $q(t) > 0$) we see this is also a zero of y , contradiction.

To prove the claim, take a point $x > a$ and close to a , so that $y(x) > 0, y'(x) > 0$. Pick an arbitrary $T > x$ and integrate by parts:

$$-\int_x^T q(t) dt = \int_x^T \frac{y''}{y} dt = \frac{y'}{y}(T) - \frac{y'}{y}(x) + \int_x^T \frac{(y')^2}{y^2} dt > -\frac{y'}{y}(x)$$

if $y'(T) > 0$ (since $y(T) > 0$ by the contradiction assumption.) But from the hypothesis we may certainly pick T large enough so that:

$$-\int_x^T q(t) dt < -\frac{y'}{y}(x),$$

contradicting the above. Thus we must have $y'(T) < 0$ for some $T > x$.

(ii) In the general case, make the change of variable in 10. above. Then $s \in [0, \infty)$, and $x(s) = y(t(s))$ is the solution of $\ddot{x} + Q(s)ds = 0$, where:

$$\int_0^\infty Q(s)ds = \int_a^\infty p(t)q(t)\frac{1}{p(t)}dt = \int_a^\infty q(t)dt = \infty.$$

From part (i), this implies $x(s)$ is oscillatory in $[0, \infty)$, so $y(t)$ is oscillatory in $[a, \infty)$.

12. *Sturm-Liouville spectral theory.* We consider the boundary-value problem on $[a, b]$ depending on a parameter $\lambda \in \mathbb{R}$, for the self-adjoint operator $L[y] = (p(t)y')' + q(t)y$:

$$L[y] + \lambda r(t)y = 0, \quad y(a) = y(b) = 0.$$

Here $r(t) > 0$ is a given ‘weight function’.

A nontrivial solution (if it exists) is unique, up to multiplication by a constant. If it exists, we say λ is an eigenvalue for L (with weight function r), and $y(t)$ is an eigenfunction.

Part (i). The eigenvalues form an increasing sequence:

$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots, \quad \text{with } \lambda_n \rightarrow \infty.$$

The eigenfunction y_{λ_n} for λ_n has exactly n zeros in $(a, b]$. In particular, y_{λ_1} has no zeros in the open interval (a, b) .

Part (ii). Eigenfunctions for different eigenvalues are orthogonal with respect to the inner product:

$$\langle f, g \rangle_r = \int_a^b f(t)g(t)r(t)dt.$$

This is an easy consequence of the identity:

$$\langle x, Ly \rangle - \langle y, Lx \rangle = p(t)W(t)|_a^b,$$

where $W = xy' - yx'$ is the Wronskian. This identity implies L (with Dirichlet boundary conditions) is self-adjoint, and the usual argument applies: if $x(t), y(t)$ are eigenfunctions for λ, μ (resp.):

$$(\lambda - \mu)\langle x, y \rangle_r = \langle x, Ly \rangle - \langle y, Lx \rangle = 0.$$

Part(iii) (Orthonormal basis.) If we normalize each function by its ‘ r -weighted L^2 norm’:

$$e_n(t) = \frac{1}{\|y_{\lambda_n}(t)\|_r} y_{\lambda_n}(t), \quad \|y_{\lambda_n}\|_r = \left[\int_a^b y_{\lambda_n}(t)^2 r(t) dt \right]^{1/2},$$

we obtain an orthonormal set of functions $\{e_n(t); n \geq 0\}$ in $[a, b]$:

$$\langle e_i(t), e_j(t) \rangle_r = 0, i \neq j; \quad \|e_n\|_r = 1 \forall n \geq 1.$$

This orthonormal set forms a ‘basis’ for the vector space $C[a, b]$ of continuous functions, in the sense of approximation:

$$\forall f \in C[a, b], \sum_{n=1}^{\infty} \langle f, e_n \rangle_r e_n(t) \rightarrow f \text{ in the } L^2 \text{ sense,}$$

meaning that the partial sums converge to f in the norm $\|\cdot\|_r$:

$$\|s_N - f\|_r \rightarrow 0 \text{ as } N \rightarrow \infty, \quad s_N(t) = \sum_{n=1}^N \langle f, e_n \rangle_r e_n(t).$$

If $f \in C^2[a, b]$ and satisfies the boundary conditions $f(a) = f(b) = 0$, we have uniform convergence:

$$\max_{t \in [a, b]} |s_N(t) - f(t)| \rightarrow 0.$$

Proof of part (i) (existence of eigenvalues.) First note that the change of independent variable described in 10. can be used to reduce the general case to the ‘pure-potential’ one, $L[y] = y'' + q(t)y$.

1. Define a function $N : R \rightarrow \mathbb{N}$ in the following way: for $\lambda \in R$, let $y_\lambda(t)$ be the solution in $[a, b]$ of the initial-value problem:

$$L_\lambda[y] = y'' + (\lambda r(t) + q(t))y = 0, \quad y(a) = 0, y'(a) = 1.$$

Note that λ is an eigenvalue (with eigenfunction $y_\lambda(t)$) exactly when $y_\lambda(b) = 0$. Define:

$$N(\lambda) = \text{number of zeros of } y_\lambda(t) \text{ in } (a, b].$$

The proof amounts to considering the properties of the function $N(\lambda)$.

2. First, $N(\lambda) = 0$ if λ is sufficiently negative. For then the potential of L_λ satisfies:

$$\lambda r(t) + q(t) \leq 0 \forall t \in [a, b].$$

(It is enough to take $\lambda \leq -\max_{t \in [a, b]}(q(t)/r(t))$.) By Sturm comparison with $y'' = 0$, we see that $y_\lambda(t)$ has no zeros other than a , so $N(\lambda) = 0$.

3. $N(\lambda)$ is nondecreasing: if $\lambda > \mu$, then $N(\lambda) \geq N(\mu)$. This follows from Sturm comparison, if we recall the potentials of L_λ and L_μ :

$$\lambda r(t) + q(t) \geq \mu r(t) + q(t), \text{ since } r(t) > 0 \text{ on } [a, b].$$

So between any two zeros of $y_\mu(t)$, there is a zero of $y_\lambda(t)$.

4. Next we show $N(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$. This follows from Sturm comparison with the constant potential ω^2 , where $\omega = n\pi/(b-a)$, with the solutions vanishing at a : $z(t) = \sin(\omega(t-a))$. This solution has exactly n zeros in $(a, b]$, at $t_i = a + i(b-a)/n$, $i = 1, 2, \dots, n$. So take λ large enough that the potential of L_λ satisfies:

$$\lambda r(t) + q(t) \geq \omega^2, \forall t \in [a, b].$$

(For instance, let $\lambda > \max_{t \in [a, b]}(|q(t)|\omega^2/r(t))$.) Then y_λ has at least n zeros in $(a, b]$, so $N(\lambda) \geq n$.

5. Suppose $N(\lambda) = n$ and $y_\lambda(b) \neq 0$. Then since all zeros of y_λ are in the open interval (a, b) and are ‘transversal’ (that is, $y'_\lambda \neq 0$ at the zeros of y_λ), for μ close enough to λ the solution $y_\mu(t)$ will have exactly n zeros in (a, b) (close to those of y_λ , and still satisfy $y_\mu(b) \neq 0$). Thus $N(\mu) = N(\lambda)$: this shows that the function N is continuous at those λ which are *not* eigenvalues, in fact constant in a neighborhood of such λ .

6. Start from a value of λ where $N(\lambda) = 0$ (so $y_\lambda(b) \neq 0$) and gradually increase λ . At some first point λ_1 we must have $y_{\lambda_1}(b) = 0$, so $N(\lambda_1) = 1$. (This uses point 4. above.) Note that $y_{\lambda_1}(t) > 0$ in the open interval (a, b) . What happens if we continue to increase λ ? We can’t have $y_\lambda(b) > 0$ for λ close to λ_1 , for this would mean the zero had moved out of the interval $(a, b]$, forcing $N(\lambda) = 0$ again, in violation of point 3. above. So the zero must ‘move in’ to the interior of the interval, so we have $y_\lambda(b) < 0$.

7. As we continue to increase λ past λ_1 , the first zero stays in the interior, but eventually we again must have $y_\lambda(b) = 0$ (say, at $\lambda = \lambda_2 > \lambda_1$), and N jumps by one again: $N(\lambda_2) = 2$. We get a second eigenfunction $y_{\lambda_2}(t)$, which has two zeros, one in the interior, the other at b . Clearly this must continue

indefinitely (since $N(\lambda) \rightarrow \infty$.) The function $N(\lambda)$ has a jump discontinuity of one exactly at the λ_n (and is right-continuous), and the eigenfunction y_{λ_n} has exactly n zeros in $(a, b]$. *This concludes the proof.*