MEASURABLE FUNCTIONS

Notation. $\mathcal{F}$ is the σ-algebra of Lebesgue-measurable subsets of $X = R^n$. Given $E \in \mathcal{F}$, $f : E \to \bar{R}$ (the extended real line) and $\alpha \in R$ we adopt the notation:

$$E\{f > \alpha\} = \{x \in E; f(x) > \alpha\} = f^{-1}((\alpha, +\infty)).$$

In this handout we adopt the notations $m(E), m^*(E), m_*(E)$ for Lebesgue measure, outer and inner Lebegue measure (resp.)

Definition. f is *measurable* if for any $\alpha \in R$, $E\{f > \alpha\}$ is in $\mathcal{F}$.

It is easy to see that this is equivalent to requiring measurability of one of the following types of sets, for any α:

$$E\{f \geq \alpha\}, \quad E\{f < \alpha\}, \quad E\{f \leq \alpha\}.$$

It is also equivalent to requiring:
(i) For each open set $A \subset R$, $f^{-1}(A) \in \mathcal{F}$,
or to requiring
(ii) For each closed set $F \subset R$, $f^{-1}(F) \in \mathcal{F}$.

Recall that an *algebra* of subsets of X is a family $\mathcal{F}$ of subsets with the properties (i) X and $\emptyset$ are in $\mathcal{F}$; (ii) If $A \in \mathcal{F}$, the complement $A^c = X \setminus A$ is also in $\mathcal{F}$; (iii) If $A, B \in \mathcal{F}$, then $A \cup B, A \cap B$ and $A \setminus B$ are also in $\mathcal{F}$.
A family of subsets of X is a σ-algebra if it is an algebra and is closed under countable union:

$$A_n \in \mathcal{F} \text{ for } n = 1, 2, \ldots \Rightarrow \bigcup_{n \geq 1} A_n \in \mathcal{F}.$$

(It follows that $\mathcal{F}$ is also closed under countable intersection.)

Given any family $\mathcal{G}$ of subsets of X, consider the intersection of all σ-algebras containing $\mathcal{G}$. This is again a σ-algebra, the σ-algebra *generated by* $\mathcal{G}$. The *Borel subsets* of R^n is the σ-algebra generated by the family of open subsets of R^n. (Note this depends only on the topology, not on any measure.) Since open sets are Lebesgue-measurable, it follows that the Borel σ-algebra is contained in the σ-algebra of Lebesgue-measurable sets. In fact we have:

Fact: $E \subset R^n$ is (Lebesgue) measurable if and only if there exists a Borel set $B \supset E$ with $m^*(B \setminus E) = 0$, if and only if there exists a Borel set $C \subset E$ with $m^*(E \setminus C) = 0$.

Problem 1. (*Preimages behave nicely.*) (i) Let $f : E \to \bar{R}$ be a function. Show that the family of subsets $\{A \subset R; f^{-1}(A) \in \mathcal{F}\}$ is a σ-algebra of subsets of R.

(ii) Show that if $f : E \to \bar{R}$ is measurable, then for every Borel set $B \subset R$ we have $f^{-1}(B) \in \mathcal{F}$.

Surprisingly, we have:

Example. There are measurable functions $f : R \to R$ and (Lebesgue) measurable sets $E \subset R$ such that $f^{-1}(E)$ is not measurable.

Let $\phi : [0,1] \to [0,1]$ be Lebesgue's singular function. The function $g(x) = x + \phi(x) : [0,1] \to [0,2]$ is invertible, and maps the standard Cantor set (which has measure zero) *onto* a set of positive measure. And it is a fact that any set of positive measure contains a non-measurable set.

In fact, the following is true: a function $f : R \to R$ maps measurable sets to measurable sets if, and only if, f maps sets of measure zero to sets of measure zero. ([Natanson, p.248, Theorem 2]).

Problem 2. (i) Let $f, g : E \to R$ be measurable. Then $\max\{f, g\}$ and $\min\{f, g\}$ are measurable. In particular, $f_+ = \max\{f, 0\}, f_- = \max\{-f, 0\}$ and $|f| = f_+ + f_-$ are measurable.

(i) Let $f : E \to R$ be measurable and $\phi : R \to R$ be continuous. Then the composition $\phi \circ f$ is measurable. (In particular $|f|^p$ (for any $p \in R$) and e^f are measurable.)

Pointwise and a.e. limits. Let $f_n, f : E \to \bar{R}$. Suppose $f_n(x) \to f(x)$ pointwise in E. Then f is measurable if each f_n is. To see this, let

$$A^k_m = E\{f_k \geq \alpha + \frac{1}{m}\}, \quad B^n_m = \bigcap_{k=n}^{\infty} A^k_n.$$

Then it is easy to see that:

$$E\{f > \alpha\} = \bigcup_{m \geq 1, n \geq 1} B^n_m,$$

and the set on the right is clearly measurable.

The same holds if we only know $f_n \to f$ a.e. in E: there is a null set $N \subset E$ such that $f_n \to f$ in $E \setminus N$. Thus f is measurable in $E \setminus N$, and also in N (since $m(N) = 0$, so f is measurable in E.

Proposition 1. (Lebesgue). Let $E \subset X$ be measurable, with $m(E) < \infty$. Suppose $f_n \to f$ a.e. in E, where f_n, f are measurable in E and finite a.e.

Then we have, for each $\sigma > 0$;

$$\lim_n m(E_n(\sigma)) = 0, \text{ where } E_n(\sigma) = \{x \in E; |f_n(x) - f(x)| \geq \sigma\}.$$

Remark: $m(E) < \infty$ is needed here: consider $f_n : R \to R, f_n(x) = 0$ for $x < n$, $f_n(x) = 1$ if $x \geq n$.

Proof. Consider the "bad sets":

$$A = E\{f = \pm\infty\}; \quad A_n = E\{f_n = \pm\infty\}; \quad B = E\{f_n \not\to f\}.$$

Then $Q = A \cup (\cup_{n\geq 1} A_n) \cup B$ has measure zero. Fixing $\sigma > 0$, let

$$R_n(\sigma) = \bigcup_{k=n}^{\infty} E_k(\sigma), \quad M = \bigcap_{n=1}^{\infty} R_n(\sigma),$$

a decreasing intersection. Since $m(E) < \infty$, we have $m(M) = \lim_n m(R_n(\sigma))$.

But it is easy to see that $M \subset Q$. So $\lim_n m(R_n(\sigma)) = 0$, which is even stronger than the claim, since $E_n(\sigma) \subset R_n(\sigma)$. This concludes the proof.

A small extension of the proof leads to a stronger result:

Egorov's theorem. Let $f_n, f : E \to R$, where $m(E) < \infty$. Then for any $\delta > 0$ we may find $F \subset E$ measurable with $m(F) \leq \delta$, so that $f_n \to f$ uniformly on $E \setminus F$.

Proof. We showed earlier that, for any $\sigma > 0$, $m(R_n(\sigma)) \to 0$. Let $(\sigma_i)_{i\geq 1}$ be any decreasing sequence of positive numbers converging to zero. Given $\delta > 0$, we find n_i so that:

$$m(R_{n_i}(\sigma_i)) < \frac{\delta}{2^i} \quad \forall i \geq 1.$$

Then letting

$$F = \bigcup_{i=1}^{\infty} R_{n_i}(\sigma_i), \quad m(F) \leq \sum_{i=1}^{\infty} m(R_{n_i}(\sigma_i)) \leq \delta,$$

it is easy to see that $f_n \to f$ uniformly in $E \setminus F$. Indeed given $\epsilon > 0$ choose $i_0 \geq 1$ so that $\sigma_{i_0} < \epsilon$. Then if $k \geq n_{i_0}$ and $x \in E \setminus F$, one verifies easily that:

$$|f_k(x) - f(x)| \leq \sigma_{i_0} < \epsilon.$$

Proposition 1 motivates the following definition.

Definition. Let $f_n, f : E \to \bar{R}$ be measurable and a.e. finite. We say f_n *converges to* f *in measure* if for all $\sigma > 0$ $\lim_n m(E_n(\sigma)) = 0$, where $E_n(\sigma) = \{x \in E; |f_n(x) - f(x)| \geq \sigma\}$.

Remark. The limit in measure of a sequence (f_n) is not unique, but any two limits coincide a.e. ([Natanson, p.97]).

We showed in Proposition 1 that pointwise convergence implies convergence in measure (for functions defined on a set of finite measure). Conversely, if $f_n \to f$ in measure, then a subsequence of (f_n) converges to f pointwise a.e.

Proposition 2. Let $f_n, f : E \to R$, where $m(E) < \infty$. Assume $f_n \to f$ in measure. Then a subsequence (f_{n_i}) converges to f a.e. in E.

Proof. With notations as before, we have $m(E_n(\sigma)) \to 0$. Let $\sigma_i > 0$ be a decreasing sequence with limit zero. For each $i \geq 1$ we may find $n_i \geq 1$ so that:

$$m(E_{n_i}(\sigma_i)) \leq \frac{1}{2^i}, \text{ and hence } m(R_k) \leq \frac{1}{2^k}, \text{ where } R_k = \bigcup_{i=k}^{\infty} E_{n_i}(\sigma_i).$$

Thus, defining:

$$N = \bigcap_{k=1}^{\infty} R_k,$$

the decreasing intersection property implies $m(N) = 0$. We claim that $f_{n_i}(x) \to f(x)$ for $x \in E \setminus N$.

To see this, let $\epsilon > 0$ be given, and let $x \in E \setminus N$. This means for some $k \geq 1$ we have: $x \in E \setminus R_k$, so for all $i \geq k$: $x \in E \setminus E_{n_i}(\sigma_i)$. Choosing $i_0 \geq k$ so that $\sigma_{i_0} < \epsilon$, we have for all $i \geq i_0$: $|f_{n_i}(x) - f(x)| < \epsilon$, as claimed.

The next result says that any given any measurable function f we may find a *closed* subset of its domain whose complement has arbitrarily small measure, so that the restriction of f to this closed set is continuous.

Luzin's theorem. Let $f : E \to R$ be a measurable function. Then for any $\delta > 0$ we may find $F \subset E$ closed so that the restriction $f_{|F}$ is continuous on F and $m(E \setminus F) \leq \delta$.

Proof. (i) Assume first $m(E) < \infty$. For each integer $k \geq 1$, we let $\{I_{k,n}\}_{n \geq 1}$ denote the partition of R into countably many intervals (left-closed, right-open) of length $1/k$, and consider the partition of E by their preimages,

$$E = \bigsqcup_{n=1}^{\infty} E_{k,n}, \quad E_{k,n} = f^{-1}(I_{k,n}).$$

For each $n \geq 1$ we may find $F_{k,n} \subset E_{k,n}$ compact, so that $m(E_{k,n} \setminus F_{k,n}) < \frac{\delta}{2^{k+n+1}}$, in particular:

$$m(E \setminus \bigsqcup_{n\geq 1}^{\infty} F_{k,n}) \leq \frac{\delta}{2^{k+1}}, \text{ so } m(E \setminus F_k) \leq \frac{\delta}{2^k}, \text{ where } F_k = \bigsqcup_{n=1}^{N_k} F_{k,n},$$

for some $N_k \geq 1$ sufficiently large. Note the F_k are closed sets, hence their intersection $F = \cap_{k\geq 1} F_k$ is also closed, and its complement in E has measure estimated by:

$$m(E \setminus F) = m(\bigcup_{k\geq 1} E \setminus F_k) \leq \delta.$$

Now define $\phi_k : F_k \to R$ by:

$$\phi_k(x) = y_{k,n} \text{ for } x \in F_{k,n},$$

where $y_{k,n} \in I_{k,n}$ is the left endpoint of the interval $I_{k,n}$. This is well-defined, since the $F_{k,n}$ for different n are disjoint. Further, since ϕ_k is constant on disjoint closed sets, it is continuous in F_k.

It is easy to see we have, for each $x \in F_k$ (in particular, for each $x \in F$:

$$|\phi_k(x) - f(x)| \leq \frac{1}{k}.$$

This shows $\phi_k \to f$ uniformly in F, hence f is continuous when restricted to F, as claimed in the statement of the theorem.

(ii) To extend this to the case when $m(E)$ is not finite, consider the partition of R^n into countably many cubes $(Q_j)_{j\geq 1}$, say of side length one. We may apply part (i) to conclude the existence of $F_j \subset E \cap Q_j$ with:

$$m[(E \cap Q_j) \setminus F_j] \leq \frac{\delta}{2^j}, \quad f_{|F_j} \text{ continuous }.$$

Since the family of cubes $\{Q_j\}$ is locally finite, the countable union of closed sets $F = \cup_{j\geq 1} F_j$ is also closed. (*Check this.*) Thus $f_{|F}$ is continuous and we estimate the measure of $E \setminus F$ by:

$$m(E \setminus F) \leq \sum_{j=1}^{\infty} m[(E \cap Q_j) \setminus F_j] \leq \delta,$$

as we wished to show.

Problem 3. Prove the converse: if $f : E \to R$ is a function with the property that for any $\delta > 0$ one may find a closed set $F \subset E$ so that the restriction $f_{|F}$ is continuous and $m(E \setminus F) < \delta$, then f is measurable.

Corollary 1. If $f : E \to R$ is measurable, for any $\delta > 0$ we may find $g_\delta : E \to R$ continuous in E, so that $m(\{x \in E; f(x) \neq g_\delta(x)\}) < \delta$. If $|f(x)| \leq K$ in E, then also $|g_\delta(x)| \leq K$ in E.

This follows from Tietze's extension theorem in Topology (which says we can extend continuous functions defined on a closed subset to continuous functions on the whole space, without increasing its sup norm): extending the function f from the closed set F given by Luzin's theorem to all of E yields g_δ.

Corollary 2. Let $f : E \to R$ be measurable. Then there exists a sequence $f_n : E \to R$ of functions continuous in E so that $f_n \to f$ a.e. in E.

Proof. Assume first $m(E) < \infty$. Letting δ_n be any sequence converging to 0 and considering the functions $f_n = g_{\delta_n}$ (continuous in E) given by Corollary 1, we see that $f_n \to f$ in measure. Thus by Proposition 2 a subsequence f_{n_j} converges to f a.e. in E.

It is easy to extend this to the case $m(E) = \infty$ (left to the reader.)