MATH 664—ALGEBRAIC TOPOLOGY II—SPRING 2022
COURSE OUTLINE AND REFERENCES

Outline. The course will consist of three interrelated parts.

Part I: Homotopy theory and connections with homology/cohomology
Part II: Characteristic classes of vector bundles and obstruction theory
Part III: Steenrod squares and applications

Applications planned: Stiefel's theorem (parallelizability of compact oriented 3-manifolds), existence of spin structures on manifolds, secondary obstructions and homotopy classes of maps in codimension one, Thom's 1954 paper on the Steenrod problem (realizability of homology classes by immersions of manifolds) and the beginnings of cobordism theory.

Main references:

1.A. Fomenko, D. Fuchs: *Homotopical Topology*—Springer-Verlag GTM no. 273—2nd ed., 2016 (you should get this.)

2. A. Hatcher, *Algebraic Topology* (Cambridge UP, 2001)

3. J. Milnor, J. Stasheff: *Characteristic Classes*—Princeton UP Annals of Math Studies no. 76 (1974)—you should get this too.

Other references:

We'll still be looking at proofs in Bredon's *Topology and Geometry* and Bott and Tu's *Differential Forms in Algebraic Topology,* when they're more transparent or complete than those found in the main sources.

An excellent source for motivation and an overview of how the subject developed is:

4.J. Dieudonne', *A History of Algebraic and Differential Topology*, 1900-1960. (Birkhäuser 1989)

Below there's a list of where the topics in the course are found in that book. It is interesting to see how certain theorems were first proved; sometimes the proofs are more geometric, and directly motivated by specific problems. So one can see how the result looked before further more general theory (algebrization) was developed. Early proofs can be too cumbersome and harder to follow than current ones, and seeing that is instructive too.

A classic source (one of the first textbooks) for most topics in the course is:
5.N. Steenrod, *The Topology of Fibre Bundles* (Princeton UP, 1951).

Very concrete, motivated and geometric. You should probably get it if it's not too hard.

Finally, towards the end of the course (time permitting) we'll try to understand as much as possible of R. Thom's 1954 paper on the "Steenrod problem" (realization of homology classes by submanifolds or immersions), which also introduced the notion of cobordism. It may be said to have ushered in the modern era in Differential Topology. It is in French, but an English translation may be found on the Web.

6.R. Thom, *Some global properties of differentiable manifolds* (*Quelques proprietés globales des varietés differentiables*, Comm. Math Helvetici 28 (1954), 17-86. 1958 translation by V. Manturov, with comments by M. Postnikov.)

(Google "Topological Library part 1 Novikov Taimanov editors")
This paper touches on many themes in the course, so one option is to "read along" throughout the term. (It is at a more advanced level, however.)

Sections from Fomenko/ Fuchs *Homotopical Topology*
PART I

8. Homotopy groups: def, commutativity, dep. on basepoint/ coverings/relative htopy groups/htopy seq of a pair

9. Fiber bundles and fibrations/ covering htopy property/Serre fibrations/weak htopy equivalence/htopy seq of a fibration/ex: htopy groups of spheres via Hopf fibration

10. Freudenthal suspension theorem/pi_n(S^n) and pi_3(S^2)/Whitehead square

11. Htopy groups and attaching cells/ 1st nontriv htopy of a CW complex/Whitehead's 1st thm: isom of htopy groups is htopy equivalence (for CW)/ Eilenberg-McLane spaces

14. Hurewicz homomorphism/Hurewicz theorem/Whitehead's 2nd thm: isomorphism of pi_n is isom of H_n.

16. Künneth formula (15.6)/ Cup product from cohomology cross product/ Hopf invariant

PART II

18. Obstruction theory. The obstruction cocycle/ cohomology and maps to K(pi, n)/Hopf's theorems on maps from X^n to the n-sphere/ obstructions to extending sections

19. vector bundles and operations/ associated bundle with fiber V(n,k) Stiefel mfd./ htopy groups of V(n,k)/Stiefel-Whitney classes/ geometric construction: classifying spaces. /Properties, axiomatic approach (Stiefel-Whitney, euler)/Differential topology applications.

PART III

29. Steenrod squares/ construction via transgression/Cartan's formula

31. Steenrod squares and Stiefel-Whitney classes/ Wu formula/Stiefel's theorem/ Steenrod. squares and secondary obstructions/nonexistence of spheroids with odd Hopf invariant.

Sections from Hatcher, *Algebraic Topology* (Ch. 4, Homotopy Theory)

Part I

4.1 Homotopy groups/ rel. homotopy groups/ exact sequence/. Ex: cone over a space. Whitehead's theorem (weak htopy equivalence implies htopy eq for CW complexes), appln of cellular approx.: pi_n(S^k) for n<k.

4.22 Wk htopy equiv implies bijection of htopy classes of maps from CW complexes.
4A: basepoints, action of pi_1.

4.2 Calculation methods. Suspension theorem, pi_n(S^n)/ Eilenberg-McLane spaces / Hurewicz theorem/ relative version

Fiber bundles, fibrations, long homotopy sequence/ Hopf fibrations/htopy lifting property/Whitehead products/ Hopf fibrations/ Stiefel and Grassmann manifolds/ Bott periodicity (statement).

4.3 Connections with cohomology. Htopy construction of cohomology (abstract proof)

4B Hopf invariant (non-geometric)/ Adams's theorem on Hopf invariant 1 (statement)

Part II
4.3 Obstruction theory (p. 415)-via Postnikov towers

4D Cohomology of fiber bundles/ Lie groups. Stiefel and Grassmann manifolds (Leray-Hirsch theorem). Euler class, Gysin sequence, Thom isomorphism.

Part III
4L Steenrod squares and powers. (abstract existence proofs)

J. Milnor, J. Stasheff, *Characteristic Classes.*
Part II

1—3 Smooth manifolds, vector bundles, constructions
4 Stiefel-Whitney classes (axiomatic)
5 Grassmann manifolds and universal bundles
6-7 Cell structure and cohomology ring for the Grassmannian
8- Existence of Stiefel-Whitney classes (using Steenrod squares)

9, 10-Euler classes of orientable bundles, Thom isomorphism theorem
11-Applications to smooth manifolds
12-Obstruction theory interp of SW and Euler classes/Gysin seq/oriented universal bundle

Sections in J. Dieudonne', *A History of Algebraic and Differential Topology, 1900-1960*
(chapters in Part 3, Homotopy and Its Relation to Homology)

Ch.2 Elementary notions and early results in homotopy theory

1: The work of H. Hopf: Brouwer's conjecture, the Hopf invariant, maps from S^(2k-1) to S^k.
2. Basic notions: extensions, retracts, homotopy type
3. Homotopy groups: Hurewicz definition, elementary properties, suspensions and loop spaces, change of basepoint.
4. Relations with homology: Hurewicz homomorphism, application to Hopf's classification (htopy classes of maps from n-dim complexes to S^n), obstruction theory
5. Relative homotopy and exact sequences/ relative Hurewicz homomorphism/ 1st Whitehead theorem (action of cont maps on homotopy and homology)
6. Homotopy of CW complexes: Hurewicz theorem on aspherical spaces/2nd Whitehead thm (wk htopy equiv of CW cplex is htopy equiv.)/CW pairs/Htopy excision/Freudenthal suspension theorem.

Ch.3 Fibrations

1.Loc. trivial fiber spaces
2. Htopy properties of fibrations: covering homotopy, fibrations v. fiber spaces, htopy exact seq of a fibration, applns to htopy groups. Classifying spaces: Whitney-Steenrod theorems.

Ch.4 Homology of Fibrations

1.Characteristic classes: Stiefel classes, Whitney's work, Pontrjagin classes
2.Gysin exact sequence

Ch. 6 Cohomology operations

1.Steenrod squares: mappings of spheres and cup products, first construction
2.New definition of Steenrod squares
4.Applications: Steenrod extension theorem (secondary obstruction), St-Wh classes, homotopy groups, nonexistence theorems.