On the Behavior of the Asymptotics of Robertson-Walker Cosmologies as a Function of the Cosmological Constant

A Dissertation
Presented for a
Master's Degree
The University of Tennessee, Knoxville

Noah Schaefferkoetter
May 2011

Acknowledgments

I would like to express my sincere gratitude to my advisor, Doctor Fernando Schwartz.

Abstract

This paper will explore the different scenarios concerning the expansion of the universe within a Robertson-Walker metric. The goal will be to establish under what conditions the Big Bang occurred as well as the situations in which a singularity does not occur. This is done by first establishing some initial assumptions; then the Einstein field equations will be reduced to three simplified equations called the Friedmann equations. Using these equations, different scenarios are examined in order to establish in what conditions the expansion of the universe compresses to a point and which of these are plausible under current knowledge.

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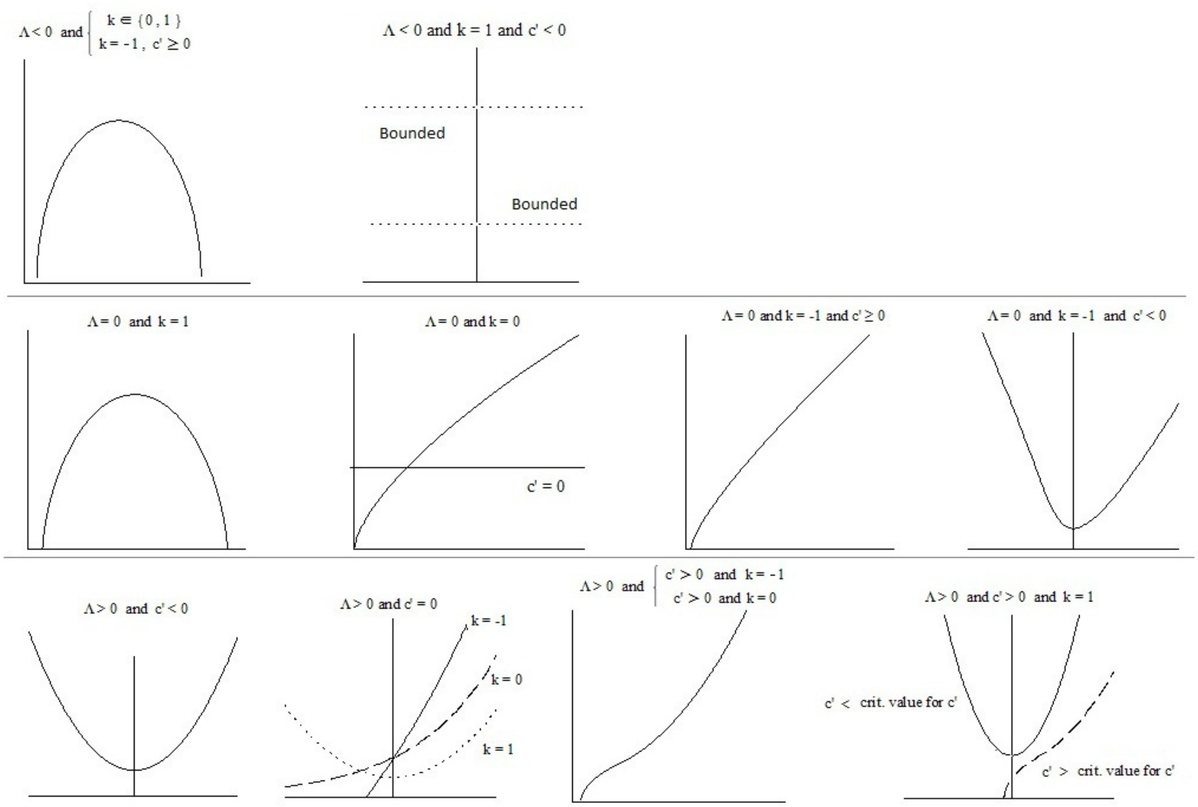
Solutions for a



Key:
Λ: cosmological constant (discussed in 1.3),
a: function depicting the "size" of the universe (introduced in chapter 2),
k: value corresponding to the geometry considered (introduced in 2.2.4),
c': a constant related to the average mass density (introduced in 3.1 and 3.2)

Chapter 1

Introduction

The event known as the Big Bang started getting serious consideration when people realized that the universe is not stagnant as once thought. Since then, there has been a great deal of work done in cosmology to show the situations in which this event occurs. This paper is an attempt to summarize the methods and results used in the model of a homogeneous and isotropic universe. Two books will be used to aid in this endeavor: *General Relativity* by Robert M. Wald and *The Large Scale Structure Of Space-Time* by S. W. Hawking and G. F. R. Ellis.

In Wald's book, he works out an almost identical set-up to the situation that will be used in this paper; as a result, many of his methods will be used. As a complement, *The Large Scale Structure Of Space-Time* by Hawking and Ellis does address the wanted results, but being highly technical, many of the steps still need to be worked out. So in addition to the original purpose of the paper, a natural secondary goal will be to unite Wald's work with Hawking and Ellis' findings.

Like every work that relays insights into a well established topic, the information that will be discussed is based upon the knowledge of the current time. With this understanding, the paper will be sectioned in three progressive parts: the Einstein field equations, the resulting metric, and finally, the possible scenarios dealing with the past and future size of the universe.

1.1 Einstein Field Equations

The Einstein field equations will be discussed first for one simple reason: of the information that will be used, the field equations can be stated with the most confidence. They create an equality between the curvature of timespace and the stress energy tensor, which, in simplified terms, is the way in which mass effects space. More importantly, these equations are used for a basis of our understanding for current cosmology.

The Einstein field equation, with the gravitational constant Λ, will be represented by the following (using geometrized units):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

(Einstein, 1915)

The metric tensor ($g_{\mu\nu}$), which will be the focus of chapter 2, is used to describe the layout of the universe. It is used to describe notions of distance, present time, and the concept of straight. In addition, many of the other terms are dependent on this metric. These include the Ricci tensor ($R_{\mu\nu}$) and the scalar curvature (R).

The Ricci tensor is best described as a measurement of the curvature of space. It compares the metric being used to a Euclidean (non-curved) space. The scalar curvature, by definition, is the trace of the Ricci curvature. Since both of these depend on the established metric, discussion of these terms will be postponed until chapter 2.

The stress-energy tensor ($T_{\mu\nu}$), on the other hand, is completely independent of the need for a metric, which leads to the next section.

1.2 Stress-energy Tensor

In order to obtain a managable formula, the same reasoning as Wald will be used. "On the cosmic scales with which we are dealing, each of the galaxies can be idealized as a 'grain of dust' "(Wald, 1984). Under the assumption that the grand scale of the universe is not greatly different than the part we can observe, it eventually follows that the best evaluation of the stress-energy tensor is the form of a perfect fluid. This leads to the equation

$$T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu) \tag{1.2}$$

(Wald, 1984)
where ρ is the average mass density, P is the pressure, and u_i is a unit vector oriented along the time axis (t).
The average mass density is just what the name implies: it is a an average of how much mass (or energy) is contained within a unit of space over the entire universe. Pressure is a measurement of the small scale movement of mass in space. Since the amount of pressure is dependent directly upon how much mass there is, there is a relationship between P and ρ. This relationship will be presented in one of two ways: the case of dust and the case of radiation (Wald, 1984).

It should also be mentioned that mass density of the universe is a function of time since the size of the universe has a potential to change over time. Instead of notating the terms $P(t)$ and $\rho(t)$, the t will be dropped for simplification and, eventually, conformity.

1.3 Λ: The Cosmological Constant

Einstein originally proposed the field equations without a cosmological constant. Near the end of his life he included the term $\Lambda g_{\mu\nu}$ in the equation in order to explain new cosmological data. Since then "Λ has been reintroduced on numerous occasions when discrepancies have arisen between theory and observations, only to be abandoned again when these discrepancies have been resolved" (Wald, 1984). Due to recent observations the cosmological constant has once again been reintroduced (Riess et al., 1998; Perlmutter et al., 1999), so the methods used by Wald will be reworked with the cosmological constant included.

Chapter 2

Robertson-Walker metric

As stated in the last chapter, both R and $R_{\mu\nu}$ are dependent on the established metric. So, in order to obtain further values, assumptions must be addressed.

In this chapter we will assume that the universe is homogeneous and isotropic. The first assumption means that there are no distinguished points in space. The second means that there are no preferred directions either.

The geometric consequence of these hypotheses is that the spacetime is forced to have a particular metric. This is the Robertson-Walker metric. In spherical coordinates (Ψ, Θ, and Φ) and Cartesian coordinates (x,y,z), we can describe the Robertson-Walker metric with the following equations:

$$ds^2 = -dt^2 + a^2(t) \begin{cases} d\Psi^2 + sin^2\Psi(d\Theta^2 + sin^2\Theta d\Phi^2) & spherical \\ dx^2 + dy^2 + dz^2 & flat \\ d\Psi^2 + sinh^2\Psi(d\Theta^2 + sin^2\Theta d\Phi^2) & hyperbolic \end{cases}$$

(Wald, 1984)
where t is a variable notating time, and $a(t)$ is a function of t whose behavior portrays the expansion/contraction of the universe. This function, $a(t)$, will be the main focus of this chapter and the next since it dictates whether or not the universe has been, or ever will achieve, a singularity ($a(t) = 0$). To do this, there is a need to find the simplest equations that depend on $a(t)$. So, the rest of the terms used in the Einstein field equations must be evaluated.

2.1 $g_{\mu\nu}$: The Metric Tensor

The first of the terms needing to be evaluated from the last chapter is the metric tensor, $g_{\mu\nu}$, which comes directly form the equations above. Noting that $ds^2 = g = \Sigma g_{\mu\nu} dx^\mu dx^\nu$, then

$$g_{\mu\nu} = \begin{cases} 0 & \mu \neq \nu \\ \xi_x & \mu = \nu = x \end{cases} \tag{2.1}$$

where $ds^2 = \Sigma \xi_x dx^2$. (For example, in the hyperbolic geometry $g_{\Psi\Psi} = a(t)^2 sinh^2(\Psi) sin^2(\Theta)$).

These values are essential to evaluating the rest of the needed terms, but before going on, there is a need to establish the Christoffel Symbols (Γ^a_{bc}) (see appendix). These will be needed to obtain the Ricci curvature tensor.

2.2 $R_{\mu\nu}$: The Ricci Curvature Tensor

Using Christoffel Symbols, the Ricci curvature tensor can be evaluated with the following:

$$R_{\mu\nu} = \Sigma_c R^c_{\mu c\nu} = \Sigma_c \left[\frac{\partial}{\partial c} \Gamma^c_{\mu\nu} - \frac{\partial}{\partial \mu} \Gamma^c_{c\nu} + \Sigma_\lambda (\Gamma^\lambda_{\mu\nu} \Gamma^c_{\lambda c} - \Gamma^\lambda_{c\nu} \Gamma^c_{\lambda\mu}) \right]$$

For the sake of efficiency, a few points will be made (which can all be verified by the equation above):

- for all geometries, if $\mu \neq \nu$ then $R_{\mu\nu} = 0$
- if $\mu \neq t$ then $g^{\mu\mu} R_{\mu\mu} = g^{xx} R_{xx} = a^{-2}(t) R_{xx}$

Since the Christoffel symbols are not the same in every geometry, three subsections will be needed to verify the general equations found at the end of the section.

2.2.1 Flat Geometry

Referring to Appendix A, the non-vanishing Christoffel symbols of the flat geometry are only the following:

$$\Gamma^x_{tx} = \Gamma^x_{xt} = \Gamma^y_{ty} = \Gamma^y_{yt} = \Gamma^z_{tz} = \Gamma^z_{zt} = \tfrac{1}{a(t)} \tfrac{da(t)}{dt}$$

$$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a(t) \tfrac{da(t)}{dt}$$

This leads to:

$$\begin{aligned} R_{tt} &= \Sigma \left[\tfrac{\partial}{\partial c} \Gamma^c_{tt} - \tfrac{\partial}{\partial t} \Gamma^c_{ct} + \Sigma(\Gamma^\lambda_{tt} \Gamma^c_{\lambda c} - \Gamma^\lambda_{ct} \Gamma^c_{\lambda t}) \right] = \tfrac{\partial}{\partial c} 0 - 3 \tfrac{\partial}{\partial t} \Gamma^x_{xt} + 0 \Gamma^c_{\lambda c} - 3 \Gamma^x_{xt} \Gamma^x_{xt} \\ &= -3 \tfrac{\partial}{\partial t} \tfrac{1}{a(t)} \tfrac{da(t)}{dt} - 3 (\tfrac{1}{a(t)} \tfrac{da(t)}{dt})^2 = 3 (\tfrac{1}{a^2(t)} (\tfrac{da(t)}{dt})^2 - \tfrac{1}{a(t)} \tfrac{d^2 a(t)}{dt^2} - 3 (\tfrac{1}{a(t)} (\tfrac{da(t)}{dt}))^2 \\ &= -3 \tfrac{1}{a(t)} \tfrac{d^2 a(t)}{dt^2} \end{aligned}$$

$$\begin{aligned} R_{xx} &= \Sigma \left[\tfrac{\partial}{\partial c} \Gamma^c_{xx} - \tfrac{\partial}{\partial x} \Gamma^c_{cx} + \Sigma(\Gamma^\lambda_{xx} \Gamma^c_{\lambda c} - \Gamma^\lambda_{cx} \Gamma^c_{\lambda x}) \right] = \tfrac{\partial}{\partial t} \Gamma^t_{xx} - \tfrac{\partial}{\partial x} 0 + 3 \Gamma^t_{xx} \Gamma^x_{tx} - 2 \Gamma^t_{xx} \Gamma^x_{tx} \\ &= \tfrac{\partial}{\partial t} a(t) \tfrac{da(t)}{dt} + \tfrac{1}{a(t)} \tfrac{da(t)}{dt} a(t) \tfrac{da(t)}{dt} = (\tfrac{da(t)}{dt})^2 + a(t) \tfrac{d^2 a(t)}{dt^2} + (\tfrac{da(t)}{dt})^2 \\ &= 2 (\tfrac{da(t)}{dt})^2 + a(t) \tfrac{d^2 a(t)}{dt^2} \end{aligned}$$

2.2.2 Spherical Geometry

To keep consistent with Appendix A, from this point on, the variables x, y, z will be used instead of the variables Ψ, Θ, Φ (respectively).

Referring to Appendix A, the non-vanishing Christoffel symbols of the spherical metric are only the following:

$$\Gamma^x_{tx} = \Gamma^x_{xt} = \Gamma^y_{ty} = \Gamma^y_{yt} = \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{1}{a(t)}\frac{da(t)}{dt}$$
$$\Gamma^y_{yx} = \Gamma^y_{xy} = \Gamma^z_{zx} = \Gamma^z_{xz} = \frac{cos(x)}{sin(x)}, \Gamma^z_{zy} = \Gamma^z_{yz} = \frac{cos(y)}{sin(y)}$$
$$\Gamma^t_{xx} = a(t)\frac{da(t)}{dt},\ \Gamma^t_{yy} = a(t)sin^2(x)\frac{da(t)}{dt},\ \Gamma^t_{zz} = a(t)sin^(x)sin^2(y)\frac{da(t)}{dt}$$
$$\Gamma^x_{yy} = -sin(x)cos(x),\ \Gamma^x_{zz} = -sin(x)cos(x)sin^2(y),\ \Gamma^y_{zz} = -sin(y)cos(y)$$

This leads to:

$$\begin{aligned} R_{tt} &= \Sigma\left[\frac{\partial}{\partial c}\Gamma^c_{tt} - \frac{\partial}{\partial t}\Gamma^c_{ct} + \Sigma(\Gamma^\lambda_{tt}\Gamma^c_{\lambda c} - \Gamma^\lambda_{ct}\Gamma^c_{\lambda t})\right] = \frac{\partial}{\partial c}0 - 3\frac{\partial}{\partial t}\Gamma^x_{xt} + 0\Gamma^c_{\lambda c} - 3\Gamma^x_{xt}\Gamma^x_{xt} \\ &= -3\frac{\partial}{\partial t}\frac{1}{a(t)}\frac{da(t)}{dt} - 3(\frac{1}{a(t)}\frac{da(t)}{dt})^2 = 3(\frac{1}{a^2(t)}(\frac{da(t)}{dt})^2 - \frac{1}{a(t)}\frac{d^2a(t)}{dt^2}) - 3(\frac{1}{a(t)}\frac{da(t)}{dt})^2 \\ &= -3\frac{1}{a(t)}\frac{d^2a(t)}{dt^2} \end{aligned}$$

$$\begin{aligned} R_{xx} &= \Sigma\left[\frac{\partial}{\partial c}\Gamma^c_{xx} - \frac{\partial}{\partial x}\Gamma^c_{cx} + \Sigma(\Gamma^\lambda_{xx}\Gamma^c_{\lambda c} - \Gamma^\lambda_{cx}\Gamma^c_{\lambda x})\right] \\ &= \frac{\partial}{\partial t}\Gamma^t_{xx} - 2\frac{\partial}{\partial x}\Gamma^y_{yx} + 3\Gamma^t_{xx}\Gamma^x_{tx} - 2\Gamma^x_{tx}\Gamma^t_{xx} - 2\Gamma^y_{yx}\Gamma^y_{yx} \\ &= \frac{\partial}{\partial t}a(t)\frac{da(t)}{dt} - 2\frac{\partial}{\partial x}\frac{cos(x)}{sin(x)} + (\frac{da(t)}{dt})^2 - 2(\frac{cos(x)}{sin(x)})^2 = \\ &= 2(\frac{da(t)}{dt})^2 + a(t)\frac{d^2a(t)}{dt^2} + 2\frac{1}{sin^2(x)} - 2(\frac{cos(x)}{sin(x)})^2 \\ &= 2(\frac{da(t)}{dt})^2 + a(t)\frac{d^2a(t)}{dt^2} + 2 \end{aligned}$$

2.2.3 Hyperbolic Geometry

Referring to Appendix A, the non-vanishing Christoffel symbols of the hyperbolic metric are only the following:

$$\Gamma^x_{tx} = \Gamma^x_{xt} = \Gamma^y_{ty} = \Gamma^y_{yt} = \Gamma^z_{tz} = \Gamma^z_{zt} = \frac{1}{a(t)}\frac{da(t)}{dt}$$
$$\Gamma^y_{yx} = \Gamma^y_{xy} = \Gamma^z_{zx} = \Gamma^z_{xz} = \frac{cosh(x)}{sinh(x)}, \Gamma^z_{zy} = \Gamma^z_{yz} = \frac{cos(y)}{sin(y)}$$
$$\Gamma^t_{xx} = a(t)\frac{da(t)}{dt},\ \Gamma^t_{yy} = a(t)sinh^2(x)\frac{da(t)}{dt},\ \Gamma^t_{zz} = a(t)sinh^2(x)sin^2(y)\frac{da(t)}{dt}$$
$$\Gamma^x_{yy} = -sinh(x)cosh(x),\ \Gamma^x_{zz} = -sinh(x)cosh(x)sin^2(y),\ \Gamma^y_{zz} = -sin(y)cos(y)$$

This leads to:

$$\begin{aligned} R_{tt} &= \Sigma\left[\frac{\partial}{\partial c}\Gamma^c_{tt} - \frac{\partial}{\partial t}\Gamma^c_{ct} + \Sigma(\Gamma^\lambda_{tt}\Gamma^c_{\lambda c} - \Gamma^\lambda_{ct}\Gamma^c_{\lambda t})\right] = \frac{\partial}{\partial c}0 - 3\frac{\partial}{\partial t}\Gamma^x_{xt} + 0\Gamma^c_{\lambda c} - 3\Gamma^x_{xt}\Gamma^x_{xt} \\ &= -3\frac{\partial}{\partial t}\frac{1}{a(t)}\frac{da(t)}{dt} - 3(\frac{1}{a(t)}\frac{da(t)}{dt})^2 = 3(\frac{1}{a^2(t)}(\frac{da(t)}{dt})^2 - \frac{1}{a(t)}\frac{d^2a(t)}{dt^2}) - 3(\frac{1}{a(t)}(\frac{da(t)}{dt}))^2 \\ &= -3\frac{1}{a(t)}(\frac{d^2a(t)}{dt^2}) \end{aligned}$$

$$
\begin{aligned}
R_{xx} &= \Sigma\left[\tfrac{\partial}{\partial c}\Gamma^{c}_{xx} - \tfrac{\partial}{\partial x}\Gamma^{c}_{cx} + \Sigma(\Gamma^{\lambda}_{xx}\Gamma^{c}_{\lambda c} - \Gamma^{\lambda}_{cx}\Gamma^{c}_{\lambda x})\right] \\
&= \tfrac{\partial}{\partial t}\Gamma^{t}_{xx} - 2\tfrac{\partial}{\partial x}\Gamma^{y}_{yx} + 3\Gamma^{t}_{xx}\Gamma^{x}_{tx} - 2\Gamma^{x}_{tx}\Gamma^{t}_{xx} - 2\Gamma^{y}_{yx}\Gamma^{y}_{yx} \\
&= \tfrac{\partial}{\partial t}a(t)\tfrac{da(t)}{dt} - 2\tfrac{\partial}{\partial x}\tfrac{cosh(x)}{sinh(x)} + (\tfrac{da(t)}{dt})^2 - 2(\tfrac{cosh(x)}{sinh(x)})^2 = \\
&= 2(\tfrac{da(t)}{dt})^2 + a(t)\tfrac{d^2a(t)}{dt^2} - 2(\tfrac{sinh^2(x)}{sinh^2(x)}) \\
&= 2(\tfrac{da(t)}{dt})^2 + a(t)\tfrac{d^2a(t)}{dt^2} - 2
\end{aligned}
$$

2.2.4 General Formulas

As seen from all of the previous work, the solutions for $R_{\mu\nu}$ can be reduced to two equations. For the sake of clarity, the following variables will be used: $a = a(t), \dot{a} = \frac{da(t)}{dt}, \ddot{a} = \frac{d^2a(t)}{dt^2}$.

$$R_{tt} = -\frac{3}{a}\ddot{a} \tag{2.2}$$

$$a^{-2}R_{xx} = 2(\frac{\dot{a}}{a})^2 + \frac{\ddot{a}}{a} + \frac{2k}{a^2} \tag{2.3}$$

where $k = 1, 0, -1$ for spherical, flat, and hyperbolic geometries respectively.

2.3 R and Simplification of the Field Equations

As stated previously, the scalar curvature of the metric is defined to be the trace of the Ricci curvature. So it is easy to see that $R = trace(Ricci) = \Sigma g^{ii}R_{ii} = g^{tt}R_{tt} + 3g^{xx}R_{xx}$, and using the evaluations from the last section

$$
\begin{aligned}
R &= g^{tt}R_{tt} + 3g^{xx}R_{xx} = -1[-\tfrac{3}{a}\ddot{a}] + 3\tfrac{1}{a^2}[2\dot{a}^2 + a\ddot{a} + 2k] = 3\tfrac{\ddot{a}}{a} + 6(\tfrac{\dot{a}}{a})^2 + 3\tfrac{\ddot{a}}{a} + 6\tfrac{k}{a^2} \\
&= 6(\tfrac{\ddot{a}}{a} + (\tfrac{\dot{a}}{a})^2 + \tfrac{k}{a^2})
\end{aligned}
$$

Noting that u_t is a unit vector that is time oriented, it follows that $u_tu_t = 1$. Alternatively, u_x is a vector tangent to a spacelike slice (which means it has no time component) projected onto a time oriented vector; so $u_xu_x = 0$. Using equations (1.1), (1.2), (2.1), (2.2), (2.3), and the previous 'R', the field equations can be simplified to the following:

$$
\begin{aligned}
3((\tfrac{\dot{a}}{a})^2 + \tfrac{k}{a^2}) - \Lambda &= [-\tfrac{3}{a}\ddot{a}] - \tfrac{1}{2}[6(\tfrac{\ddot{a}}{a} + (\tfrac{\dot{a}}{a})^2 + \tfrac{k}{a^2})][-1] + \Lambda[-1] = 8\pi[\rho u_tu_t + P(-1 + u_tu_t)] \\
&= 8\pi[\rho(1) + P(-1+1) = 8\pi\rho
\end{aligned}
$$

$$
\begin{aligned}
-(\tfrac{\dot{a}}{a})^2 - 2\tfrac{\ddot{a}}{a} - \tfrac{k}{a^2} + \Lambda &= [2(\tfrac{\dot{a}}{a})^2 + \tfrac{\ddot{a}}{a} + \tfrac{2k}{a^2}] - \tfrac{1}{2}[6(\tfrac{\ddot{a}}{a} + (\tfrac{\dot{a}}{a})^2 + \tfrac{k}{a^2})][1] + \Lambda[1] \\
&= 8\pi[\rho u_xu_x + P(a^{-2}g_{xx} + u_xu_x)] = 8\pi[\rho(0) + P(1+0)] = 8\pi P
\end{aligned}
$$

2.4 Friedmann Equations

Finally, in order to make future work a little easier, we will derive what are known as the Friedmann equations named after Alexander Friedmann. Using his equations, Friedmann was actually the first to derive the Robertson-Walker metric; the full name of which is the Friedmann-Lematre-Robertson-Walker metric. Unfortunately, not many took notice of his work until it was reproduced independently after his death in 1925.

These three equations are the most condensed solution of the function a when considering unknown geometries, cosmological constants, and relationships between pressure and mass density within the Robertson-Walker metric. With the equations that have already been derived above, the simplifications occur by solving for a form of $\dot{a}$ then, through substitution, solving for a form of $\ddot{a}$.

$$3([\Lambda - \tfrac{k}{a^2} - 8\pi P - 2\tfrac{\ddot{a}}{a}] + \tfrac{k}{a^2}) - \Lambda = 8\pi\rho \Rightarrow 3\tfrac{\ddot{a}}{a} = \Lambda - 4\pi(\rho + 3P)$$

$$-(\tfrac{\dot{a}}{a})^2 - 2[\tfrac{1}{3}(\Lambda - 4\pi(\rho + 3P))] - \tfrac{k}{a^2} + \Lambda = 8\pi P \Rightarrow 3(\tfrac{\dot{a}}{a})^2 = \Lambda + 8\pi\rho - \tfrac{3k}{a^2}$$

It then follows, by differentiating and substitution, that:

$$-\tfrac{\dot{a}}{a}24\pi(\rho + P) + \tfrac{6\dot{a}k}{a^3} = 2(\tfrac{\dot{a}}{a}([\Lambda - 4\pi(\rho + 3P)] - [\Lambda + 8\pi\rho - \tfrac{3k}{a^2}]) = (3(2\tfrac{\dot{a}}{a}\tfrac{a\ddot{a} - \dot{a}^2}{a^2})$$
$$= \tfrac{d}{dt}(3(\tfrac{\dot{a}}{a})^2) = \tfrac{d}{dt}(\Lambda + 8\pi\rho - \tfrac{3k}{a^2}) = 8\pi\dot{\rho} + \tfrac{6\dot{a}k}{a^3}$$

So, in summary, the Friedmann Equations are:

$$3\frac{\ddot{a}}{a} = \Lambda - 4\pi(\rho + 3P) \tag{2.4}$$

$$3(\frac{\dot{a}}{a})^2 = \Lambda + 8\pi\rho - \frac{3k}{a^2} \tag{2.5}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \tag{2.6}$$

and will be used as a starting point while solving for a in the next chapter.

Chapter 3

Values of Λ

Notice that within the Friedmann equations from the end of last chapter, there are 4 variables:

- ρ (average mass density)
- P (pressure)
- k (type of geometry)
- Λ (cosmological constant)

It is clear that if these are not restricted, it would be absurd to try to find a solution for a.

Fortunately, P depends on ρ in one of two ways: dust or radiation (Wald, 1984). When understanding these, it should be noted that mass is vary rare in the cosmos, so the situation in which mass collides is very unusual and may be dismissed for large scale analysis. Dust is the situation in which mass is stable and interacts gravitationally, or $P = 0$. Radiation, on the other hand, is the situation in which mass has random movement; this causes there to be an outward force (pressure) which combines with the influence of gravity. This can be explained with the notation $P = \frac{1}{3}\rho$. Since these are the only two cases that will be considered, it is reasonable to use them to separate the chapter. Additionally, when introducing each of these sections, it will be explained more throughly how mass (and therefore pressure) can be further refined to a constant c'.

k was introduced in section 2.2.4 where it was defined to reference the three different geometries: spherical, flat, and hyperbolic. These are represented as $k = 1, 0$, or -1 respectively. Thus, the paper will need to be divided further using these values to grant a finer restriction on a.

As for the cosmological constant, Λ, this (like the soon to be introduced c') is not a discrete variable, and therefore, does not have a finite amount of values to choose from. This issue can be avoided by analyzing what is needed from the Friedmann Equations: the sign. Remembering that the main goal of the text is to find when $a = 0$, we can use basic calculus to find the needed information by looking at the sign of $\dot{a}$ and $\ddot{a}$. With this understanding, the constant Λ (and for the most part c') has three different scenarios: positive, 0, and negative.

Finally, it should be noted that it is safe to claim that $a > 0$. The reasoning comes from the understanding that a is continuous, and at the current time, $t_0, a(t_0) \neq 0$. Supposing at some point $a = 0$, then this is a singularity of space and thus is no longer being considered 'the universe'. With this in mind, a will be restricted to the domain $t \in (t_m, t_M)$ where $t_m = max\{x : x < t_0 \text{ and } a(x) = 0, -\infty\}$ and $t_M = min\{x : x > t_0 \text{ and } a(x) = 0, \infty\}$. So for all $t \in (t_m, t_M), a(t)a(t_0) > 0 \Rightarrow a(t) = |a(t)|$.

3.1 Dust $P = 0$

Using equation (2.6) and integration, if $P = 0$ then

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \Rightarrow \rho = \frac{c'}{a^3}$$

where c' is a constant that is introduced during integration. Since $a > 0$ then $sign(c') = sign(\rho)$.

With this, we can revise the Friedmann Equations to the following:

$$3\frac{\ddot{a}}{a} = \Lambda - 4\pi\frac{c'}{a^3} \tag{3.1}$$

$$3(\frac{\dot{a}}{a})^2 = \Lambda + 8\pi\frac{c'}{a^3} - \frac{3k}{a^2} \tag{3.2}$$

3.1.1 $\Lambda < 0$

Claim: In the dust case, if $\Lambda < 0$, then as long as the universe is not hyperbolic or the universe has a nonnegative mass density, there was a Big Bang that began the universe, where $a(t_m) = 0$ and there will eventually be a time in the future in which the universe will collapse, $a(t_M) = 0$.

Proof:
Case 1: (spherical and flat geometries [or $k = 1, 0$])

Note from equation (3.2), $\Lambda + 8\pi\frac{c'}{a^3} - \frac{3k}{a^2} \geq 0$ which implies $0 < -\frac{a^3}{8\pi}(\Lambda - \frac{3k}{a^2}) \leq c'$. Now, by using this inequality in equation (3.1), it is clear that $3\frac{\ddot{a}}{a} \leq \Lambda < 0$, and (using the knowledge that a is positive) it follows that a is concave down and $\dot{a}$ is decreasing.

Suppose that $\dot{a} \neq 0$. So, $|\dot{a}|$ is bounded below by a real value $a_\infty > 0$ and $\dot{a}$ is decreasing. Using the definition of bounded, as $t \to \infty$ (or $-\infty$), $|\dot{a}| \to a_\infty$, which means that $\ddot{a} \to 0$. Using the inequality $3\frac{\ddot{a}}{a} < \Lambda < 0$, it follows that $\ddot{a} < \frac{a\Lambda}{3} < 0$, and using the squeeze theorem, $a \to 0$. This causes a contradiction for the following reason: if $a \to 0$ then $\dot{a}^2 = a^2\Lambda + \frac{8\pi c'}{a} - 3k \to \infty \neq a_\infty^2$. So there is some time, t_*, so that $\dot{a}(t_*) = 0$.

So, a is concave down and has a max value of $a(t)$, which means that $a(t_m) = a(t_M) = 0$.

Case 2: (hyperbolic geometry [or $k = -1$] and non-negative mass density $c' \geq 0$)

This is similar to case 1 in the sense that $c' \geq 0$ gives $3\frac{\ddot{a}}{a} \leq \Lambda < 0$ which implies a is concave down and $\dot{a}$ is decreasing. It then follows by the same reasoning that there exists a t_* so that $\dot{a}(t_*) = 0$. Therefore, a is decreasing and obtains a max value, which results in $a(t_m) = a(t_M) = 0$.

Q.E.D.

Otherwise: (hyperbolic geometry [or $k = -1$] and negative mass density $c' < 0$)

The difference from this case to the rest is: the only positive term in the equation for $3(\frac{\dot{a}}{a})^2$ is $\frac{3}{a^2}$. This and the fact that $3(\frac{\dot{a}}{a})^2 \geq 0$ results in the following:

$$3 \geq -a^2\Lambda - 8\pi\frac{c'}{a} > -8\pi\frac{c'}{a}$$

So, $a > -8\pi\frac{c'}{3} > 0$, and by similar reasoning, $a^2 < -\frac{3}{\Lambda}$. This mean that the expansion of the universe is bounded above and below by some $k > 0$, so there will never be any point at which $a = 0$.

In addition, it is meaningful to note that in the case of Λ is too big or c' is too small, then the universe couldn't exist. So there are some restrictions if certain unlikely events were to come up.

3.1.2 $\Lambda = 0$

It should be noted that Wald actually came up with a distinct solution to each one of the situations within the $\Lambda = 0$ case. For the sake of consistency, the cases are still going to be checked to verify that there are no conditions that vary from his findings. Only after verification will these equations be stated.

Case 1: Spherical (or $k = 1$)

Referring to (3.2), $\Lambda + 8\pi\frac{c'}{a^3} - \frac{3k}{a^2} = 8\pi\frac{c'}{a^3} - \frac{3}{a^2} \geq 0$, which means $8\pi\frac{c'}{3} \geq a$, so $c' > 0$. Consequently, using (3.1), $3\ddot{a}a^2 = -4\pi c' < 0$.

Suppose that $\dot{a} \neq 0$. It follows that there exists a real number a_∞ so that as $t \to \infty$ (or $-\infty$), $\dot{a} \to a_\infty$. That leads to $\ddot{a} \to 0$, and consequently, $a \to \infty$ (by similar reasoning to 3.1.1 case 1). This contradicts the previous result of $8\pi\frac{c'}{3} \geq a$, which means the supposition is wrong, and there must exist a t_* so that $\dot{a}(t_*) = 0$. By the same reasoning from last section, the universe began and will end in a singularity.

This is consistent with the form that was stated by Wald:

$$a = \frac{4\pi c'}{3}(1 - cos(\eta))$$

$$t = \frac{4\pi c'}{3}(\eta - sin(\eta))$$

(Wald, 1984)

Case 2: Flat (or $k = 0$) and $c' \neq 0$

Since $0 \leq 3(\frac{\dot{a}}{a})^2 = 8\pi\frac{c'}{a^3}$ then $c' > 0$. Supposing that there exists a t_* so that $\dot{a}(t_*) = 0$, then by (3.2) $c' = 0$ or $a(t_*) = \infty$. Now since a is continuous and $c' \neq 0$, then there is no alternative to $t_* \in \{-\infty, \infty\}$. So a is constantly increasing or constantly decreasing–faster as $a \to 0$–which implies that either:

- the universe began in a singularity and will continue to grow forever, or
- the universe had no start but at all times it was previously bigger and will eventually collapse to a singularity.

Referring again to Wald, when $c' \neq 0$, the form that a will take in this case is similar to:

$$(6\pi c')^{\frac{1}{3}} t^{\frac{2}{3}}$$

(Wald, 1984)

Case 3: flat (or $k = 0$) and $c' = 0$

In the case of $c' = 0$, simply by observing equation (3.2), it follows that a is constant. Thus it has no expansion and will always be the stagnant universe that Einstein originally believed it to be.

Case 4: hyperbolic (or $k = -1$)

By (3.2), $0 \leq \frac{8\pi}{3}\frac{c'}{a} + 1$, which means $a \geq -\frac{8\pi c'}{3}$. Now supposing that $c' \geq 0$, then $\dot{a}^2 = 8\pi 3\frac{c'}{a} + 1 \geq 1$, so $\dot{a}^2$ is bounded below by some positive value a_*. This means that a has a slope that never gets too close to zero, and therefore, a takes the same form as case 2. As long as $c' \neq 0$, this case follows the equations obtained in Wald's book:

$$a = \frac{4\pi c'}{3}(cosh\eta - 1)$$

$$t = \frac{4\pi c'}{3}(sinh\eta - \eta)$$

(Wald, 1984)

Alternatively, if $c' < 0$, then $a \geq -\frac{8\pi c'}{3}$, so a is bounded below by some positive value a_*. Noting that $\frac{-4\pi c'}{3a_*^2} > 0$, then as $a \to a_*$, $\ddot{a}$ does not approach 0, so a cannot level off at any constant value. By this reasoning, the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

3.1.3 $\Lambda > 0$

With $\Lambda > 0$, the implications are not as easy to come by. Since $3(\frac{\dot{a}}{a})^2 - \Lambda$ can potentially be negative, the other constants are not as easy to restrict. So, the cases will be divided up a little differently.

Case 1: $(c' < 0)$

In this case, it is easy to see (using 3.1) that $\ddot{a} > \Lambda > 0$. Since $lim_{a \to 0}\dot{a}^2 = -\infty < 0 < \infty = lim_{a \to \infty}\dot{a}^2$, then it is impossible for a to be too close to 0, and there must exist some $a_* \in (0, \infty)$ such that $a_*^3\Lambda + 8\pi c' + a_* 3k = 0$. So a_* is a lower bound of a since all $a < a_*$ will cause $3\dot{a}^2 = a^2\Lambda + \frac{8\pi c'}{a} + 3k < 0$, which is obviously false. Noting that a is concave up and obtains a min at a_*, then it follows that the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

Case 2: Vacuum (or $c' = 0$)

First, note that in this case, $\Lambda > 0$ implies $\ddot{a} > 0$ with $\ddot{a} \to 0$ if and only if $a \to 0$.

If the geometry of the universe is hyperbolic ($k = -1$), then it is clear that $3\dot{a}^2 = a^2\Lambda + 3 > 3$. So, the universe will never have a slope (with absolute value) less than $3^{\frac{1}{2}}$. Therefore, the universe takes one of the following two forms:

- the universe began in a singularity and will continue to grow forever, or
- the universe had no start but at all times it was previously bigger and will eventually collapse to a singularity.

Note that if the geometry is flat, then the same argument cannot be made since $3k = 0$. Nevertheless, since $a > 0$ leads to $\dot{a}^2 = a^2\Lambda > 0$, then for all positive values E, there is some t so that $a(t) < E$. So, for practical reasons the universe will eventually get within a threshold of a singularity, and it follows the same form.

Alternatively, if the universe has a spherical geometry ($k = 1$), then (using (3.2)) $0 \le \Lambda - \frac{3}{a^2}$, and consequently, $a^2 \ge \frac{3}{\Lambda} > 0$. So, a would reach a *min* value at $a = (\frac{3}{\Lambda})^{\frac{1}{2}} > 0$, and since $\ddot{a} = a\Lambda \ge (3\Lambda)^{\frac{1}{2}} > 0$, it follows that the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

Case 3: $(c' > 0$ in a hyperbolic or flat geometry$)$

Note that $\frac{d}{da}3\dot{a}^2 = 2a\Lambda - \frac{8\pi c'}{a^2}$. So, when $\frac{d}{da}\dot{a} = \ddot{a} = 0$, it follows that $3\dot{a}^2 \geq 3\Lambda^{\frac{1}{3}}(4\pi c')^{\frac{2}{3}} - 3k$. Now, under the assumption that $k \in \{0, -1\}$, then a is always increasing or always decreasing by a rate of at least $(3\Lambda^{\frac{1}{3}}(4\pi c')^{\frac{2}{3}})^{\frac{1}{2}}$. So a will once again fall into one of two cases:

- the universe began in a singularity and will continue to grow forever, or
- the universe had no start but at all times it was previously bigger and will eventually collapse to a singularity.

Case 4: ($c' > 0$ in a spherical geometry)

In this case, there is both the possibility to change concavity and the possibility of changing slope. Therefore, it must be determined where these occur. So noting that $\ddot{a} = 0$ when $a = a_* = (\frac{4\pi c'}{\Lambda})^{\frac{1}{3}}$, then $\dot{a}^2$ cannot change slope unless $\dot{a}^2 = \Lambda^{\frac{1}{3}}(4\pi c')^{\frac{2}{3}} - 1$.

Using this reasoning, if $c' > \frac{1}{\Lambda^{\frac{1}{2}}4\pi}$, then $\dot{a}^2$ changes slope at a_* since $a_*^2\Lambda + \frac{8\pi c'}{a_*} - 3 > 0$ so $\dot{a}^2$ has a positive lower bound, which means a is either constantly increasing or decreasing by some set amount. So if $c' > \frac{1}{\Lambda^{\frac{1}{2}}4\pi}$, then this yields the same results as case 3.

On the other hand, if $c' < \frac{1}{\Lambda^{\frac{1}{2}}4\pi}$, then $a \neq a_*$ since $\dot{a}^2 \geq 0 > (\Lambda)^{\frac{1}{3}}(4\pi c')^{\frac{2}{3}} - 1$. So there exists some lower bound of $\ddot{a}$, a_0, so that $\ddot{a} \geq a_0 > 0$. This results in a always being concave up, and as before, since $\ddot{a}$ is bounded below by a positive value a_0, then it follows that the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

3.2 Radiation ($\rho = \frac{1}{3}P$)

Much of this section will be the same as the previous section, but there are some significant differences. All of the changes will result directly from the change in the Friedmann equations. Since $P \neq 0$, the following results occur:

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \Rightarrow \rho = \frac{c'}{a^4}$$

As in the previous section, this leads to:

$$3\frac{\ddot{a}}{a} = \Lambda - 8\pi\frac{c'}{a^4} \quad (3.3)$$

$$3(\frac{\dot{a}}{a})^2 = \Lambda + 8\pi\frac{c'}{a^4} - \frac{3k}{a^2} \quad (3.4)$$

3.2.1 $\Lambda < 0$

Claim: In the dust case, if $\Lambda < 0$, then as long as the universe is not hyperbolic or the universe has a nonnegative mass density, there was a Big Bang that began the universe, where $a(t_m) = 0$ and there will eventually be a time in the future in which the universe will collapse, $a(t_M) = 0$.

Proof:
Case 1: (spherical and flat geometries [or $k = 1, 0$])

Note from equation (3.4), $\Lambda + 8\pi\frac{c'}{a^4} - \frac{3k}{a^2} \geq 0$ which implies $0 \leq -\frac{a^4}{8\pi}(\Lambda - \frac{3k}{a^2}) \leq c'$. Now, by using this inequality in equation (3.3), it is clear that $3\frac{\ddot{a}}{a} < \Lambda < 0$, and (using the knowledge that a is positive) it follows that a is concave down and $\dot{a}$ is decreasing.

Suppose that $\dot{a} \neq 0$. So, $|\dot{a}|$ is bounded below by a real value $a_\infty > 0$ and $\dot{a}$ is decreasing. Using the definition of bounded, as $t \to \infty$ (or $-\infty$), $|\dot{a}| \to a_\infty$, which means that $\ddot{a} \to 0$. Using the inequality $3\frac{\ddot{a}}{a^2} < \Lambda < 0$, it follows that $\ddot{a} < \frac{a\Lambda}{3} < 0$, and using the squeeze theorem, $a \to 0$. This causes a contradiction for the following reason: if $a \to 0$ then $\dot{a}^2 = a^2\Lambda + \frac{8\pi c'}{a^2} - 3k \to \infty \neq a_\infty^2$. So there is some time, t_*, so that $\dot{a}(t_*) = 0$.

So, a is concave down and has a max value of $a(t)$, which means that $a(t_m) = a(t_M) = 0$.

Case 2: (hyperbolic geometry [or $k = -1$] and non-negative mass density $c' \geq 0$)

This is similar to case 1 in the sense that $c' \geq 0$ gives $3\frac{\ddot{a}}{a} \leq \Lambda < 0$ which implies a is concave down and $\dot{a}$ is decreasing. It then follows by the same reasoning that there exists a t_* so that $\dot{a}(t_*) = 0$. Therefore, a is decreasing and obtains a max value, which results in $a(t_m) = a(t_M) = 0$.

Q.E.D.

Otherwise: (hyperbolic geometry [or $k = -1$] and negative mass density $c' < 0$)

The difference from this case to the rest is: the only positive term in the equation for $3(\frac{\dot{a}}{a})^2$ is $\frac{3}{a^2}$. This and the fact that $3(\frac{\dot{a}}{a})^2 \geq 0$ results in the following:

$$3 \geq -a^2\Lambda - 8\pi\frac{c'}{a^2} > -8\pi\frac{c'}{a^2}$$

So, $a^2 > -8\pi\frac{c'}{3} > 0$, and by similar reasoning, $a^2 < -\frac{3}{\Lambda}$. This mean that the expansion of the universe is bounded above and below by some $k > 0$, so there will never be any point at which $a = 0$.

In addition, it is meaningful to note that in the case of Λ is too big or c' is too small, then the universe couldn't exist. So there are some restrictions if certain unlikely events were to come up.

3.2.2 $\Lambda = 0$

Similar to the dust case, Wald actually came up with a distinct solution to each of the situations within $\Lambda = 0$, although these will not be used without verification.

Case 1: Spherical (or $k = 1$)

Referring to (3.4), $\Lambda + 8\pi\frac{c'}{a^4} - \frac{3k}{a^2} = 8\pi\frac{c'}{a^4} - \frac{3}{a^2} \geq 0$, which means $8\pi\frac{c'}{3} \geq a^2$, so $c' > 0$. Consequently, using (3.3), $3\ddot{a}a^3 = -8\pi c' < 0$.

Suppose that $\dot{a} \neq 0$. It follows that there exists a real number a_∞ so that as $t \to \infty$ (or $-\infty$), $\dot{a} \to a_\infty$. That leads to $\ddot{a} \to 0$, and consequently, $a \to \infty$ (by similar reasoning to 3.2.1 case 1). This contradicts the previous result of $8\pi\frac{c'}{3} \geq a^2$, which means the supposition is wrong, and there must exist a t_* so that $\dot{a}(t_*) = 0$. By the same reasoning from last section, the universe began and will end in a singularity.

This is consistent with the form that was stated by Wald:

$$\sqrt{c'}[1 - (1 - \frac{t}{\sqrt{c'}})^2]^{\frac{1}{2}}$$

(Wald, 1984)

Case 2: Flat (or $k = 0$) and $c' \neq 0$

Since $0 \leq 3(\frac{\dot{a}}{a})^2 = 8\pi\frac{c'}{a^4}$ then $c' > 0$. Supposing that there exists a t_* so that $\dot{a}(t_*) = 0$, then by (3.4) $c' = 0$ or $a(t_*) = \infty$. Now since a is continuous and $c' \neq 0$, then there is no alternative to $t_* \in \{-\infty, \infty\}$. So a is constantly increasing or constantly decreasing–faster as $a \to 0$–which implies that either:

- the universe began in a singularity and will continue to grow forever, or
- the universe had no start but at all times it was previously bigger and will eventually collapse to a singularity.

Referring again to Wald, when $c' \neq 0$, the form that a will take in this case is similar to:

$$(4\pi c')^{\frac{1}{4}} t^{\frac{1}{2}}$$

(Wald, 1984)

Case 3: flat (or $k = 0$) and $c' = 0$

In the case of $c' = 0$, simply by observing equation (3.4), it follows that a is constant. Thus it has no expansion and will always be the stagnant universe that Einstein originally believed it to be.

Case 4: hyperbolic (or $k = -1$)

By (3.4), $0 \leq \frac{8\pi}{3} \frac{c'}{a^2} + 1$, which means $a^2 \geq -\frac{8\pi c'}{3}$. Now supposing that $c' \geq 0$, then $\dot{a}^2 = 8\pi 3 \frac{c'}{a^2} + 1 \geq 1$, so $\dot{a}^2$ is bounded below by some positive value a_*. This means that a has a slope that never gets too close to zero, and therefore, a takes the same form as case 2. As long as $c' \neq 0$, this case follows the equations obtained in Wald's book:

$$\sqrt{c'}[(1 + \frac{t}{\sqrt{c'}})^2 - 1]^{\frac{1}{2}}$$

(Wald, 1984)

Alternatively, if $c' < 0$, then $a^2 \geq -\frac{8\pi c'}{3}$, so a is bounded below by some positive value a_*. Noting that $\frac{-8\pi c'}{3a_*^3} > 0$, then as $a \to a_*$, $\ddot{a}$ does not approach 0, so a cannot level off at any constant value. By this reasoning, the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

3.2.3 $\Lambda > 0$

With $\Lambda > 0$, the implications are not as easy to come by. Since $3(\frac{\dot{a}}{a})^2 - \Lambda$ can potentially be negative, the other constants are not as easy to restrict. So, the cases will be divided up a little differently.

Case 1: ($c' < 0$)

In this case, it is easy to see (using 3.3) that $\ddot{a} > \Lambda > 0$. Since $lim_{a\to 0}\dot{a}^2 = -\infty < 0 < \infty = lim_{a\to\infty}\dot{a}^2$, then it is impossible for a to be too close to 0, and there must exist some $a_* \in (0, \infty)$ such that $a_*^4\Lambda + 8\pi c' + a_*^2 3k = 0$. So a_* is a lower bound of a since all $a < a_*$ will cause $3\dot{a}^2 = a^2\Lambda + \frac{8\pi c'}{a^2} + 3k < 0$, which is obviously false. Noting that a is concave up and obtains a min at a_*, then it follows that the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

Case 2: Vacuum (or $c' = 0$)

First, note that in this case, $\Lambda > 0$ implies $\ddot{a} > 0$ with $\ddot{a} \to 0$ if and only if $a \to 0$.

If the geometry of the universe is hyperbolic ($k = -1$), then it is clear that $3\dot{a}^2 = a^2\Lambda + 3 > 3$. So, the universe will never have a slope (with absolute value) less than $3^{\frac{1}{2}}$. Therefore, the universe takes one of the following two forms:

- the universe began in a singularity and will continue to grow forever, or
- the universe had no start but at all times it was previously bigger and will eventually collapse to a singularity.

Note that if the geometry is flat, then the same argument cannot be made since $3k = 0$. Nevertheless, since $a > 0$ leads to $\dot{a}^2 = a^2\Lambda > 0$, then for all positive values E, there is some t so that $a(t) < E$. So, for practical reasons the universe will eventually get within a threshold of a singularity, and it follows the same form.

Alternatively, if the universe has a spherical geometry ($k = 1$), then (using (3.4)) $0 \leq \Lambda - \frac{3}{a^2}$, and consequently, $a^2 \geq \frac{3}{\Lambda} > 0$. So, a would reach a *min* value at $a = (\frac{3}{\Lambda})^{\frac{1}{2}} > 0$, and since $\ddot{a} = a\Lambda \geq (3\Lambda)^{\frac{1}{2}} > 0$, it follows that the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

Case 3: ($c' > 0$ in a hyperbolic or flat geometry)

Note that $\frac{d}{da}3\dot{a}^2 = 2a\Lambda - \frac{16\pi c'}{a^2}$. So, when $\frac{d}{da}\dot{a} = \ddot{a} = 0$, it follows that $3\dot{a}^2 \geq 2(\Lambda 8\pi c')^{\frac{1}{2}} - 3k$. Now, under the assumption that $k \in \{0, -1\}$, then a is always increasing or always decreasing by a rate of at least $(2(\Lambda 8\pi c')^{\frac{1}{2}})^{\frac{1}{2}}$. So a will once again fall into one of two cases:

- the universe began in a singularity and will continue to grow forever, or
- the universe had no start but at all times it was previously bigger and will eventually collapse to a singularity.

Case 4: ($c' > 0$ in a spherical geometry)

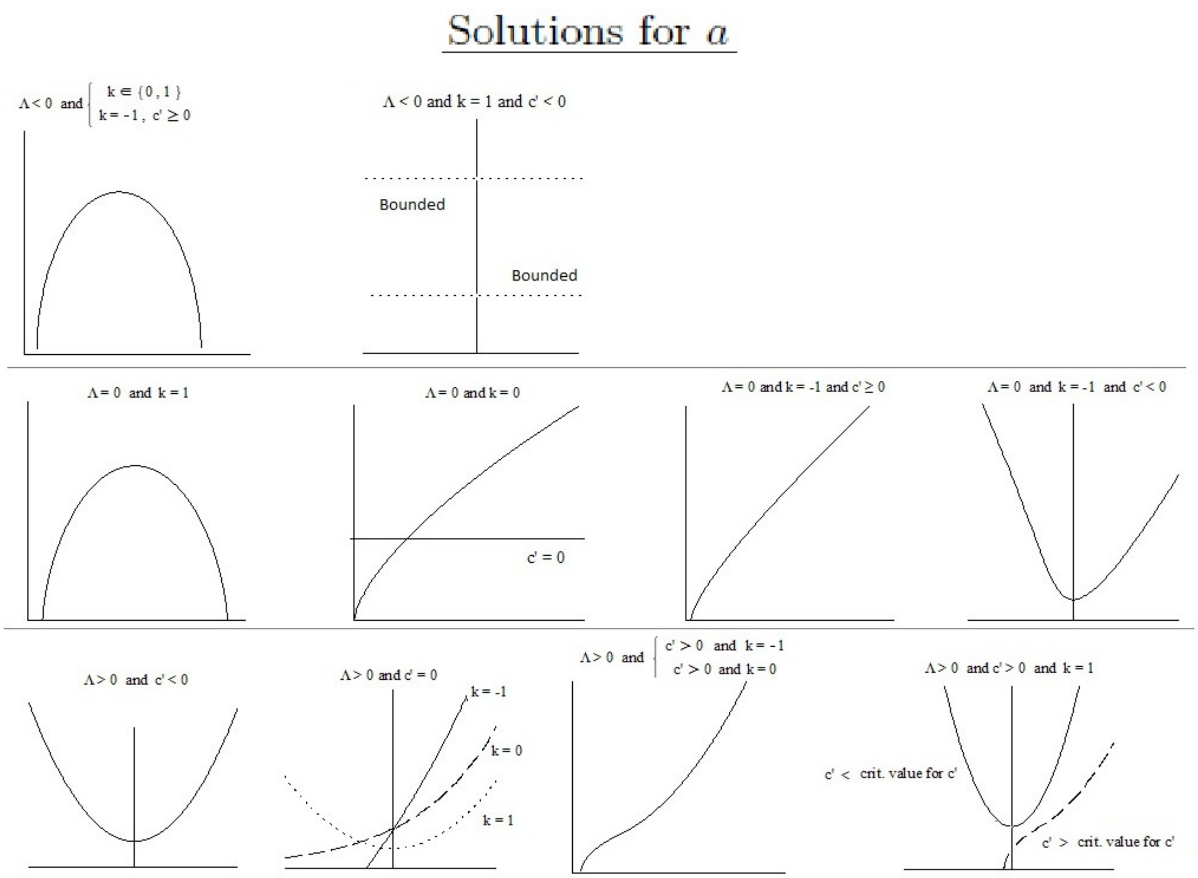
In this case, there is both the possibility to change concavity and the possibility of changing slope. Therefore, it must be determined where these occur. So noting that $\ddot{a} = 0$ when $a = a_* = (\frac{8\pi c'}{\Lambda})^{\frac{1}{4}}$, then $\dot{a}^2$ cannot change slope unless $3\dot{a}^2 = 2(\Lambda 8\pi c')^{\frac{1}{2}} - 3$.

Using this reasoning, if $c' > \frac{9}{32\Lambda\pi}$, then $\dot{a}^2$ changes slope at a_* since $a_*^2\Lambda + \frac{8\pi c'}{a_*^2} - 3 > 0$ so $\dot{a}^2$ has a positive lower bound, which means a is either constantly increasing or decreasing by some set amount. So if $c' > \frac{9}{32\Lambda\pi}$, then this yields the same results as case 3.

On the other hand, if $c' < \frac{9}{32\Lambda\pi}$, then $a \neq a_*$ since $3\dot{a}^2 \geq 0 > 2(\Lambda 8\pi c')^{\frac{1}{2}} - 3$. So there exists some lower bound of $\ddot{a}$, a_0, so that $\ddot{a} \geq a_0 > 0$. This results in a always being concave up, and as before, since $\ddot{a}$ is bounded below by a positive value a_0, then it follows that the size of the universe–in both the past and the future–will approach infinity and never achieve a singularity.

Chapter 4

Recent Knowledge and Summary



Now that all the previous cases have been determined, it is clear what restrictions mathematics has imposed on the expansion of the universe under our assumptions of isotropy and homogeneity. The graphs above give a general form of each case with the exception that in the cases of a singularity, the reflection is also a mathematical possibility. What has not been considered yet is recent knowledge that has surfaced based on the observations of the universe. It should be pointed out that if we consider Hubble's constant ($\frac{\dot{a}}{a}$) (Hawking and Ellis, 1973), which is known to be positive at the current time, then our results match up directly with the results found in the book by Hawking and Ellis. Namely, when $c' > 0$ and Λ is negative, zero, or not too large of a value, then a reaches a singularity (Hawking and Ellis, 1973).

Furthermore, there has been a recent observation of redshifts showing that the universe is expanding at an increasing rate and Λ is currently evaluated as being greater than 0 (Riess et al., 1998; Perlmutter et al., 1999). When these new restrictions are taken into account, the universe can be any case in which $\Lambda > 0$.

From this, it can be concluded that the only conditions that would cause the Big Bang not to exist is the following:

- the universe has a hyperbolic geometry ($k = 1$), and
- $c' >$ critical value (discussed in 3.1.3 case 4 and 3.2.3 case 4).

The reason that these bits of information were not introduced earlier was for the sake of future progress; new information is always being found and the purpose of this paper was to fill in gaps for knowledge that, as of today, is taken for granted or misunderstood. This is not to say that this paper contains all possibilities, but, if new information is found, hopefully this paper will aid in creating a smoother transition from the knowledge that we have now.

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Appendices

Appendix A

Solutions to Christoffel Symbols

It has been shown that $\Gamma^i_{jk} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{j\sigma}}{\partial k} + \frac{\partial g_{k\sigma}}{\partial j} - \frac{\partial g_{jk}}{\partial \sigma})$ (Wald, 1984). Now we will evaluate each Christoffel Symbol depending on the metric in question.

A.1 Flat

This metric gives us the following matrix

$$\begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \alpha^2(t) & 0 & 0 \\ 0 & 0 & \alpha^2(t) & 0 \\ 0 & 0 & 0 & \alpha^2(t) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha^2(t)} & 0 & \\ 0 & 0 & \frac{1}{\alpha^2(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha^2(t)} \end{pmatrix}^{-1} = \begin{pmatrix} g_{tt} & g_{xt} & g_{yt} & g_{zt} \\ g_{tx} & g_{xx} & g_{yx} & g_{zx} \\ g_{ty} & g_{yx} & g_{yy} & g_{zy} \\ g_{tz} & g_{xz} & g_{yz} & g_{zz} \end{pmatrix}^{-1}$$

which we use to get:

For $i \in \{t, x, y, z\}$:

$$\Gamma^i_{tt} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{t\sigma}}{\partial t} + \frac{\partial g_{t\sigma}}{\partial t} - \frac{\partial g_{tt}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ti}}{\partial t} + \frac{\partial g_{ti}}{\partial t} - \frac{\partial g_{tt}}{\partial i}) = \frac{1}{2}g^{ii}(\frac{\partial (0)}{\partial t} + \frac{\partial (0)}{\partial t} - \frac{\partial (-1)}{\partial i})$$
$$= \frac{1}{2}g^{ii}(0) = 0$$

For $i \in \{x, y, z\}$:

$$\Gamma^t_{ti} = \Gamma^t_{it} = \frac{1}{2}\Sigma g^{t\sigma}(\frac{\partial g_{t\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial t} - \frac{\partial g_{ti}}{\partial \sigma}) = \frac{1}{2}g^{tt}(\frac{\partial g_{tt}}{\partial i} + \frac{\partial g_{it}}{\partial t} - \frac{\partial g_{ti}}{\partial t}) = \frac{1}{2}g^{tt}(0 + 0 - 0) = 0$$

$$\Gamma^i_{it} = \Gamma^i_{ti} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{t\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial t} - \frac{\partial g_{ti}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ti}}{\partial i} + \frac{\partial g_{ii}}{\partial t} - \frac{\partial g_{ti}}{\partial i}) = \frac{1}{2}\frac{1}{\alpha^2(t)}(0 + \frac{\partial \alpha^2(t)}{\partial t} - 0)$$
$$= \frac{2\alpha(t)}{2\alpha^2(t)}\frac{\partial \alpha(t)}{\partial t} = \frac{1}{\alpha(t)}\frac{\partial \alpha(t)}{\partial t}$$

$$\Gamma^t_{ii} = \frac{1}{2}\Sigma g^{t\sigma}(\frac{\partial g_{i\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial i} - \frac{\partial g_{ii}}{\partial \sigma}) = \frac{1}{2}g^{tt}(\frac{\partial g_{it}}{\partial i} + \frac{\partial g_{it}}{\partial i} - \frac{\partial g_{ii}}{\partial t}) = \frac{1}{2}(-1)(0+0-\frac{\partial \alpha^2(t)}{\partial t}) = \alpha(t)\frac{\partial \alpha(t)}{\partial t}$$

For i,j$\in \{x,y,z\}$

$\Gamma^i_{jj} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{j\sigma}}{\partial j} + \frac{\partial g_{j\sigma}}{\partial j} - \frac{\partial g_{jj}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ji}}{\partial j} + \frac{\partial g_{ji}}{\partial j} - \frac{\partial g_{jj}}{\partial i}) = \frac{1}{2}g^{ii}(0 + 0 - \frac{\partial \alpha^2(t))}{\partial j} = 0$

$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{i\sigma}}{\partial j} + \frac{\partial g_{j\sigma}}{\partial i} - \frac{\partial g_{ji}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ji}}{\partial i} + \frac{\partial g_{ii}}{\partial j} - \frac{\partial g_{ij}}{\partial i}) = \frac{1}{2}g^{ii}(0 + \frac{\partial \alpha^2(t)}{\partial j} - 0) = 0$

For $i, j, k \in \{t, x, y, z\}, i \neq j, j \neq k$, and $k \neq i$

$\Gamma^i_{jk} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{j\sigma}}{\partial k} + \frac{\partial g_{k\sigma}}{\partial j} - \frac{\partial g_{jk}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ji}}{\partial k} + \frac{\partial g_{ki}}{\partial j} - \frac{\partial g_{jk}}{\partial i}) = \frac{1}{2}g^{ii}(\frac{\partial(0)}{\partial k} + \frac{\partial(0)}{\partial j} - \frac{\partial(0)}{\partial i}) = 0$

A.2 Spherical

This metric gives us the following matrix

$$\begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{y} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \alpha^2(t) & 0 & 0 \\ 0 & 0 & \alpha^2(t)sin^2(x) & 0 \\ 0 & 0 & 0 & \alpha^2(t)sin^2(x)sin^2(y) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha^2(t)} & 0 & \\ 0 & 0 & \frac{1}{\alpha^2(t)sin^2(x)} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha^2(t)sin^2(x)sin^2(y)} \end{pmatrix}^{-1} = \begin{pmatrix} g_{tt} & g_{xt} & g_{yt} & g_{zt} \\ g_{tx} & g_{xx} & g_{yx} & g_{zx} \\ g_{ty} & g_{yx} & g_{yy} & g_{zy} \\ g_{tz} & g_{xz} & g_{yz} & g_{zz} \end{pmatrix}^{-1}$$

which we use to get:

For $i \in \{t, x, y, z\}$:

$\Gamma^i_{tt} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{t\sigma}}{\partial t} + \frac{\partial g_{t\sigma}}{\partial t} - \frac{\partial g_{tt}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ti}}{\partial t} + \frac{\partial g_{ti}}{\partial t} - \frac{\partial g_{tt}}{\partial i}) = \frac{1}{2}g^{ii}(\frac{\partial(0)}{\partial t} + \frac{\partial(0)}{\partial t} - \frac{\partial(-1)}{\partial i})$
$= \frac{1}{2}g^{ii}(0) = 0$

For $i \in \{x, y, z\}$ and $\Psi(x, y) = 1, sin^2(x), sin^2(x)sin^2(y)$ for $i = x, y, z$ respectively:

$\Gamma^t_{ti} = \Gamma^t_{it} = \frac{1}{2}\Sigma g^{t\sigma}(\frac{\partial g_{t\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial t} - \frac{\partial g_{ti}}{\partial \sigma}) = \frac{1}{2}g^{tt}(\frac{\partial g_{tt}}{\partial i} + \frac{\partial g_{it}}{\partial t} - \frac{\partial g_{ti}}{\partial t}) = \frac{1}{2}g^{tt}(0 + 0 - 0) = 0$

$\Gamma^i_{it} = \Gamma^i_{ti} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{t\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial t} - \frac{\partial g_{ti}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ti}}{\partial i} + \frac{\partial g_{ii}}{\partial t} - \frac{\partial g_{ti}}{\partial i})$
$= \frac{1}{2}\frac{1}{\alpha^2(t)\Psi(x,y)}(0 + \frac{\partial \alpha^2(t)\Psi(x,y)}{\partial t} - 0) = \frac{2\alpha(t)\Psi(x,y)}{2\alpha^2(t)\Psi(x,y)}\frac{\partial \alpha(t)}{\partial t} = \frac{1}{\alpha(t)}\frac{\partial \alpha(t)}{\partial t}$

$\Gamma^t_{ii} = \frac{1}{2}\Sigma g^{t\sigma}(\frac{\partial g_{i\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial i} - \frac{\partial g_{ii}}{\partial \sigma}) = \frac{1}{2}g^{tt}(\frac{\partial g_{it}}{\partial i} + \frac{\partial g_{it}}{\partial i} - \frac{\partial g_{ii}}{\partial t}) = \frac{1}{2}(-1)(0 + 0 - \frac{\partial \alpha^2(t)\Psi(x,y)}{\partial t})$
$= \frac{2\alpha(t)\Psi(x,y)}{2}\frac{\partial \alpha(t)}{\partial t} = \alpha(t)\Psi(x, y)\frac{\partial \alpha(t)}{\partial t}$

For $i \in \{y, z\}$where $\Psi(y), \Psi(t, y)$ is the appropriate function depending on i

$\Gamma^i_{ix} = \Gamma^i_{xi} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{x\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial x} - \frac{\partial g_{xi}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{xi}}{\partial i} + \frac{\partial g_{ii}}{\partial x} - \frac{\partial g_{xi}}{\partial i})$
$= \frac{1}{2}\frac{1}{sin^2(x)\Psi(t,y)}(0 + \frac{\partial sin^2(x)\Psi(t,y)}{\partial x} - 0) = \frac{2sin(x)cos(x)\Psi(t,y)}{2sin^2(x)\Psi(t,y)} = \frac{cos(x)}{sin(x)}$

$\Gamma^x_{ii} = \frac{1}{2}\Sigma g^{x\sigma}(\frac{\partial g_{i\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial i} - \frac{\partial g_{ii}}{\partial \sigma}) = \frac{1}{2}g^{xx}(\frac{\partial g_{ix}}{\partial i} + \frac{\partial g_{ix}}{\partial i} - \frac{\partial g_{ii}}{\partial x}) = \frac{1}{2}(\frac{1}{\alpha^2(t)})(0 + 0 - \frac{\partial \alpha^2(t)sin^2(x)\Psi(y)}{\partial x})$
$= -\frac{2\alpha^2(t)sin(x)cos(x)\Psi(y)}{2\alpha^2(t)} = -sin(x)cos(x)\Psi(y)$

$$\Gamma^z_{zy} = \Gamma^z_{yz} = \frac{1}{2}\Sigma g^{z\sigma}(\frac{\partial g_{y\sigma}}{\partial z} + \frac{\partial g_{z\sigma}}{\partial y} - \frac{\partial g_{yz}}{\partial \sigma}) = \frac{1}{2}g^{zz}(\frac{\partial g_{yz}}{\partial z} + \frac{\partial g_{zz}}{\partial y} - \frac{\partial g_{yz}}{\partial z})$$
$$= \frac{1}{2}\frac{1}{\alpha^2(t)sin^2(x)sin^2(y)}(0 + \frac{\partial \alpha^2(t)sin^2(x)sin^2(y)}{\partial x} - 0) = \frac{2\alpha^2(t)sin^2(x)sin(y)cos(y)}{2\alpha^2(t)sin^2(x)sin^2(y)} = \frac{cos(y)}{sin(y)}$$
$$\Gamma^y_{zz} = \frac{1}{2}\Sigma g^{y\sigma}(\frac{\partial g_{z\sigma}}{\partial z} + \frac{\partial g_{z\sigma}}{\partial z} - \frac{\partial g_{zz}}{\partial \sigma}) = \frac{1}{2}g^{yy}(\frac{\partial g_{zy}}{\partial z} + \frac{\partial g_{yz}}{\partial z} - \frac{\partial g_{zz}}{\partial y})$$
$$= \frac{1}{2}(\frac{1}{\alpha^2(t)sin^2(x)})(0 + 0 - \frac{\partial \alpha^2(t)sin^2(x)sin^2(y)}{\partial y}) = -\frac{2\alpha^2(t)sin^2(x)sin(y)cos(y)}{2\alpha^2(t)sin^2(x)} = -sin(y)cos(y)$$

If $i = z$ then $j \in \{t, x, y\}$. If $i = y$ then $j \in \{t, x\}$. If $i = x$ then $j = t$

$$\Gamma^j_{ji} = \Gamma^j_{ij} = \frac{1}{2}\Sigma g^{j\sigma}(\frac{\partial g_{i\sigma}}{\partial j} + \frac{\partial g_{j\sigma}}{\partial i} - \frac{\partial g_{ij}}{\partial \sigma}) = \frac{1}{2}g^{jj}(\frac{\partial g_{ij}}{\partial j} + \frac{\partial g_{jj}}{\partial i} - \frac{\partial g_{ij}}{\partial j}) = \frac{1}{2}g^{jj}(0 + 0 - 0) = 0$$
$$\Gamma^i_{jj} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{j\sigma}}{\partial j} + \frac{\partial g_{j\sigma}}{\partial j} - \frac{\partial g_{jj}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ji}}{\partial j} + \frac{\partial g_{ji}}{\partial j} - \frac{\partial g_{jj}}{\partial i}) = \frac{1}{2}g^{ii}(0 + 0 - 0) = 0$$

For $i, j, k \in \{t, x, y, z\}, i \neq j, j \neq k$, and $k \neq i$ or $i = j = k$

$$\Gamma^i_{jk} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{j\sigma}}{\partial k} + \frac{\partial g_{k\sigma}}{\partial j} - \frac{\partial g_{jk}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ji}}{\partial k} + \frac{\partial g_{ki}}{\partial j} - \frac{\partial g_{jk}}{\partial i}) = \frac{1}{2}g^{ii}(0 + 0 - 0) = 0$$

A.3 Hyperbolic

This metric gives us the following matrix

$$\begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \alpha^2(t) & 0 & 0 \\ 0 & 0 & \alpha^2(t)sinh^2(x) & 0 \\ 0 & 0 & 0 & \alpha^2(t)sinh^2(x)sin^2(y) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha^2(t)} & 0 & \\ 0 & 0 & \frac{1}{\alpha^2(t)sinh^2(x)} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha^2(t)sinh^2(x)sin^2(y)} \end{pmatrix}^{-1} = \begin{pmatrix} g_{tt} & g_{xt} & g_{yt} & g_{zt} \\ g_{tx} & g_{xx} & g_{yx} & g_{zx} \\ g_{ty} & g_{yx} & g_{yy} & g_{zy} \\ g_{tz} & g_{xz} & g_{yz} & g_{zz} \end{pmatrix}^{-1}$$

which we use to get:

For $i \in \{t, x, y, z\}$:

$$\Gamma^i_{tt} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{t\sigma}}{\partial t} + \frac{\partial g_{t\sigma}}{\partial t} - \frac{\partial g_{tt}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ti}}{\partial t} + \frac{\partial g_{ti}}{\partial t} - \frac{\partial g_{tt}}{\partial i}) = \frac{1}{2}g^{ii}(\frac{\partial(0)}{\partial t} + \frac{\partial(0)}{\partial t} - \frac{\partial(-1)}{\partial i})$$
$$= \frac{1}{2}g^{ii}(0) = 0$$

For $i \in \{x, y, z\}$ and $\Psi(x, y) = 1, sinh^2(x), sinh^2(x)sin^2(y)$ for $i = x, y, z$ respectively:

$$\Gamma^t_{ti} = \Gamma^t_{it} = \frac{1}{2}\Sigma g^{t\sigma}(\frac{\partial g_{t\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial t} - \frac{\partial g_{ti}}{\partial \sigma}) = \frac{1}{2}g^{tt}(\frac{\partial g_{tt}}{\partial i} + \frac{\partial g_{it}}{\partial t} - \frac{\partial g_{ti}}{\partial t}) = \frac{1}{2}g^{tt}(0 + 0 - 0) = 0$$
$$\Gamma^i_{it} = \Gamma^i_{ti} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{t\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial t} - \frac{\partial g_{ti}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ti}}{\partial i} + \frac{\partial g_{ii}}{\partial t} - \frac{\partial g_{ti}}{\partial i})$$
$$= \frac{1}{2}\frac{1}{\alpha^2(t)\Psi(x,y)}(0 + \frac{\partial \alpha^2(t)\Psi(x,y)}{\partial t} - 0) = \frac{2\alpha(t)\Psi(x,y)}{2\alpha^2(t)\Psi(x,y)}\frac{\partial \alpha(t)}{\partial t} = \frac{1}{\alpha(t)}\frac{\partial \alpha(t)}{\partial t}$$
$$\Gamma^t_{ii} = \frac{1}{2}\Sigma g^{t\sigma}(\frac{\partial g_{i\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial i} - \frac{\partial g_{ii}}{\partial \sigma}) = \frac{1}{2}g^{tt}(\frac{\partial g_{it}}{\partial i} + \frac{\partial g_{it}}{\partial i} - \frac{\partial g_{ii}}{\partial t}) = \frac{1}{2}(-1)(0 + 0 - \frac{\partial \alpha^2(t)\Psi(x,y)}{\partial t})$$
$$= \frac{2\alpha(t)\Psi(x,y)}{2}\frac{\partial \alpha(t)}{\partial t} = \alpha(t)\Psi(x, y)\frac{\partial \alpha(t)}{\partial t}$$

For $i \in \{y, z\}$where $\Psi(y), \Psi(t,y)$ is the appropriate function depending on i

$$\Gamma^i_{ix} = \Gamma^i_{xi} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{x\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial x} - \frac{\partial g_{xi}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{xi}}{\partial i} + \frac{\partial g_{ii}}{\partial x} - \frac{\partial g_{xi}}{\partial i})$$
$$= \frac{1}{2}\frac{1}{sinh^2(x)\Psi(t,y)}(0 + \frac{\partial sinh^2(x)\Psi(t,y)}{\partial x} - 0) = \frac{2sinh(x)cosh(x)\Psi(t,y)}{2sinh^2(x)\Psi(t,y)} = \frac{cosh(x)}{sinh(x)}$$
$$\Gamma^x_{ii} = \frac{1}{2}\Sigma g^{x\sigma}(\frac{\partial g_{i\sigma}}{\partial i} + \frac{\partial g_{i\sigma}}{\partial i} - \frac{\partial g_{ii}}{\partial \sigma}) = \frac{1}{2}g^{xx}(\frac{\partial g_{ix}}{\partial i} + \frac{\partial g_{ix}}{\partial i} - \frac{\partial g_{ii}}{\partial x}) = \frac{1}{2}(\frac{1}{\alpha^2(t)})(0+0-\frac{\partial \alpha^2(t)sinh^2(x)\Psi(y)}{\partial x})$$
$$= -\frac{2\alpha^2(t)sinh(x)cosh(x)\Psi(y)}{2\alpha^2(t)} = -sinh(x)cosh(x)\Psi(y)$$
$$\Gamma^z_{zy} = \Gamma^z_{yz} = \frac{1}{2}\Sigma g^{z\sigma}(\frac{\partial g_{y\sigma}}{\partial z} + \frac{\partial g_{z\sigma}}{\partial y} - \frac{\partial g_{yz}}{\partial \sigma}) = \frac{1}{2}g^{zz}(\frac{\partial g_{yz}}{\partial z} + \frac{\partial g_{zz}}{\partial y} - \frac{\partial g_{yz}}{\partial z})$$
$$= \frac{1}{2}\frac{1}{\alpha^2(t)sinh^2(x)sin^2(y)}(0+\frac{\partial \alpha^2(t)sinh^2(x)sin^2(y)}{\partial x}-0) = \frac{2\alpha^2(t)sinh^2(x)sin(y)cos(y)}{2\alpha^2(t)sinh^2(x)sin^2(y)} = \frac{cos(y)}{sin(y)}$$
$$\Gamma^y_{zz} = \frac{1}{2}\Sigma g^{y\sigma}(\frac{\partial g_{z\sigma}}{\partial z} + \frac{\partial g_{z\sigma}}{\partial z} - \frac{\partial g_{zz}}{\partial \sigma}) = \frac{1}{2}g^{yy}(\frac{\partial g_{zy}}{\partial z} + \frac{\partial g_{yz}}{\partial z} - \frac{\partial g_{zz}}{\partial y})$$
$$= \frac{1}{2}(\frac{1}{\alpha^2(t)sinh^2(x)})(0+0-\frac{\partial \alpha^2(t)sinh^2(x)sin^2(y)}{\partial y}) = -\frac{2\alpha^2(t)sinh^2(x)sin(y)cos(y)}{2\alpha^2(t)sinh^2(x)} = -sin(y)cos(y)$$

If $i = z$ then $j \in \{t, x, y\}$. If $i = y$ then $j \in \{t, x\}$. If $i = x$ then $j = t$

$$\Gamma^j_{ji} = \Gamma^j_{ij} = \frac{1}{2}\Sigma g^{j\sigma}(\frac{\partial g_{i\sigma}}{\partial j} + \frac{\partial g_{j\sigma}}{\partial i} - \frac{\partial g_{ij}}{\partial \sigma}) = \frac{1}{2}g^{jj}(\frac{\partial g_{ij}}{\partial j} + \frac{\partial g_{jj}}{\partial i} - \frac{\partial g_{ij}}{\partial j}) = \frac{1}{2}g^{jj}(0+0-0) = 0$$
$$\Gamma^i_{jj} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{j\sigma}}{\partial j} + \frac{\partial g_{j\sigma}}{\partial j} - \frac{\partial g_{jj}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ji}}{\partial j} + \frac{\partial g_{ji}}{\partial j} - \frac{\partial g_{jj}}{\partial i}) = \frac{1}{2}g^{ii}(0+0-0) = 0$$

For $i, j, k \in \{t, x, y, z\}, i \neq j, j \neq k$, and $k \neq i$ or $i = j = k$

$$\Gamma^i_{jk} = \frac{1}{2}\Sigma g^{i\sigma}(\frac{\partial g_{j\sigma}}{\partial k} + \frac{\partial g_{k\sigma}}{\partial j} - \frac{\partial g_{jk}}{\partial \sigma}) = \frac{1}{2}g^{ii}(\frac{\partial g_{ji}}{\partial k} + \frac{\partial g_{ki}}{\partial j} - \frac{\partial g_{jk}}{\partial i}) = \frac{1}{2}g^{ii}(0+0-0) = 0$$

Vita

As of 2011, I am enrolled as a graduate student at the University of Tennessee in Knoxville. I was admitted into U.T. as an undergraduate in 2002 after being awarded the Appalachian Scholarship and graduated in 2006 with a B.S. in mathematics. After perusing a career in teaching, I returned to continue my education with an aim to obtain a higher degree. I am expecting to graduate this year with a M.S. in mathematics.