Self-Force on a Classical Point Charge

Robert M. Wald

(with Sam Gralla and Abe Harte, Phys.Rev. **D80** 024031 (2009); arXiv:0905.2391)

The Basic Issue

An accelerating point charge radiates electromagnetic energy and momentum. Therefore, there must be some "back reaction" force on the charge associated with this radiation. This can be understood as resulting from the effects of the point charge's own electromagnetic field on itself. However, the electromagnetic field of a point charge is singular at the charge itself, and its self-electromagnetic energy is infinite, so it is not obvious how to do a proper self-force or energy conservation argument. For more than a century, there has been much discussion and debate about this issue.

Classical Electrodynamics as Taught in Courses

At least 95% of what is taught in electrodynamics courses at all levels focuses on the following two separate problems: (i) Given a distribution of charges and/or currents, find the electric and magnetic fields (i.e., solve Maxwell's equations with given source terms). (ii) Given the electric and magnetic fields, find the motion of a point charge (possibly with an electric and/or magnetic dipole moment) by solving the Lorentz force equation (possibly with additional dipole force terms).

Problem (i): Since Maxwell's equations are linear, it makes perfectly good mathematical sense to allow distributional sources, such as a point charge (i.e., a

3-dimensional δ-function, non-zero only on a timelike worldline). Indeed, the general solution for continuous sources can be found by "superposition" of the solution obtained for 4-dimensional δ-function sources (i.e., the Green's function), so even if one is interested in continuous sources, it is extremely useful to consider Maxwell's equations for distributional sources.

Problem (ii): There are no mathematical difficulties in solving for the motion of a point charge in a given electromagnetic field.

Thus, for most of what is done in courses in electromagnetism, one could take the view that electrodynamics is formulated at a fundamental level in terms of point charges, as normally is done in elementary electrodynamics courses. Continuous charge distributions could be viewed as a limit of many small point charges.

However, the situation changes dramatically when one tries to consistently solve problems (i) and (ii) simultaneously. The coupled system of Maxwell's equations and the motion of charges is nonlinear, and point charges simply don't make mathematical sense! The difficulties associated with the fact that the field of a point charge is singular at the location of the charge and that the point charge has infinite self-energy—which can be ignored when solving (i) or (ii) separately—cannot be ignored in the coupled Maxwell-motion problem. This kind situation is very familiar to people who work in general relativity: Einstein's equation is nonlinear, and the notion of a "point mass" makes no sense.

How Has the Self-Force Problem Been Analyzed?

One approach—starting with Abraham (1903) and Lorentz (1904)—is to consider a finite-sized body for which the coupled Maxwell-motion problem is well defined. A simple model for a charged body, such as a rigid shell, is usually considered. One then makes approximations corresponding to a small size of the body and derives equations of motion. Among the problems with this approach are: (1) Rigid motion is not consistent with special relativity for a body undergoing non-uniform acceleration. (2) It is far from obvious that the motion of the body is independent of the matter model. (3) Since one can't take the limit of zero size without introducing infinite self-energy and other problems, it isn't obvious what the range of validity of the approximations are.

Another approach—starting with Dirac (1938)—is to work with a point charge and analyze conservation of energy-momentum in a small "worldtube" surrounding the point charge. One encounters singular expressions in this approach, but these can be regularized/renormalized. However, although these regularizations are relatively natural looking, it is far from obvious that they are correct. Also, if a point charge really had finite total energy as assumed here, it would really have to have infinitely negative "bare mass".

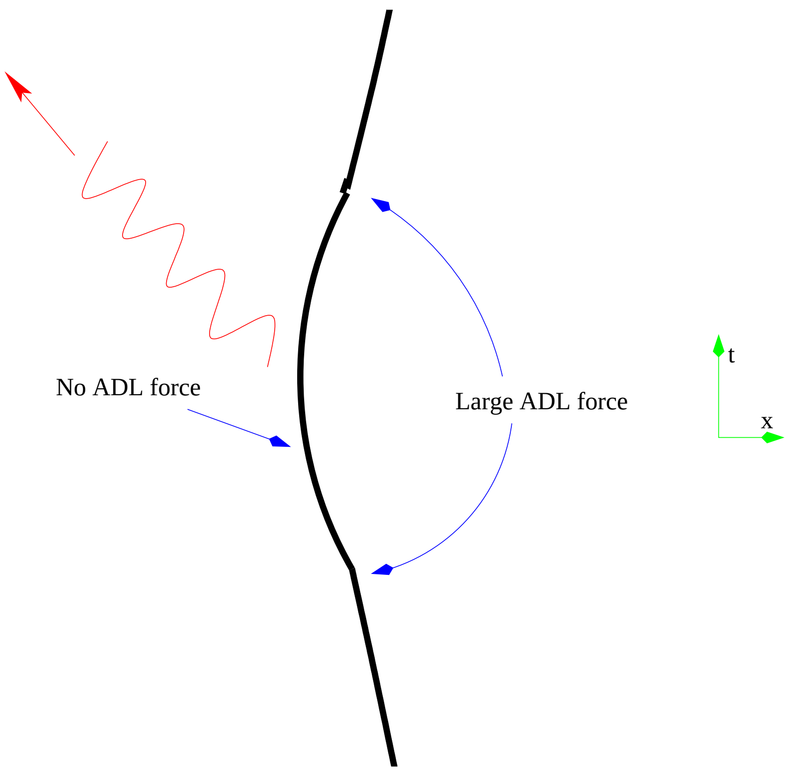
Both approaches lead to a "radiation reaction" or "self-force", known as the Abraham-Lorentz-Dirac (ALD) force, which, in the non-relativistic limit, takes the form

$$\vec{F} = \frac{2}{3} q^2 \frac{d\vec{a}}{dt} \ .$$

This results in serious difficulties. The equation $\vec{F} = m\vec{a}$ is now third order in time, so to specify initial conditions, one needs to give not only the initial position and velocity but also the initial acceleration. Worse yet, even with no external field, this equation admits "runaway" solutions, where the position of the charge grows exponentially with time. The issue of how exclude/eliminate this runaway behavior has been debated extensively during the past century.

Another Strange Feature of the ALD Force

As is well known, a uniformly accelerating charge radiates energy to infinity. However, a uniformly accelerating charge does not have any associated ALD force.



If it doesn't take any extra work to keep a charge in uniformly acceleration, doesn't one get the radiated energy "for free"? In fact, it is not difficult to show that if the charge begins and ends in inertial motion, then the total work done overcoming the ALD force is equal to the total energy radiated. However, it might seem that energy is only conserved "on average" and that one can get local violations of conservation of energy. This is difficult to analyze on account of the infinte self-energy of a point charge.

Our Approach

We will use only Maxwell's equations and (exact, local) conservation of energy and momentum. In this way, we can be certain that any general results we derive are independent of the composition of the body, and that energy and momentum are always exactly conserved.

Maxwell's equations:

$$\nabla^\nu F_{\mu\nu} = 4\pi J_\mu$$

$$\nabla_{[\mu} F_{\nu\rho]} = 0$$

Conservation of energy and momentum:

$$\nabla^\mu T_{\mu\nu} = 0, \qquad T_{\mu\nu} = T^M_{\mu\nu} + T^{EM}_{\mu\nu}$$

Stress energy tensor of the electromagnetic field:

$$T^{EM}_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

The Key Idea of Our Approach

If we do not take a limit of zero size for the body, we will have to consider the internal dynamics of the body. The motion will be complicated and will depend on the details of the composition of the body. No simple, universal equations can arise. However, if we take the usual point particle limit (zero size at fixed charge and mass), we will encounter the serious problems associated with a singular electromagnetic field and infinite self-energy that have plagued analyses for the past century.

Our approach: Consider a modified point particle limit, wherein the body shrinks to zero size in an asymptotically self-similar manner, so that not only the size of the body goes to zero, but its charge and mass also go to zero in proportion to its size. Note that the body disappears completely at $\lambda = 0$, but, like the Cheshire Cat in Alice in Wonderland, its "smile" (i.e, the worldline that the body shrinks down to) remains behind. This "smile" provides the leading order description of motion; by working perturbatively off the "smile", we obtain the self-force (and dipole) corrections to motion.

How the Charge Density Scales

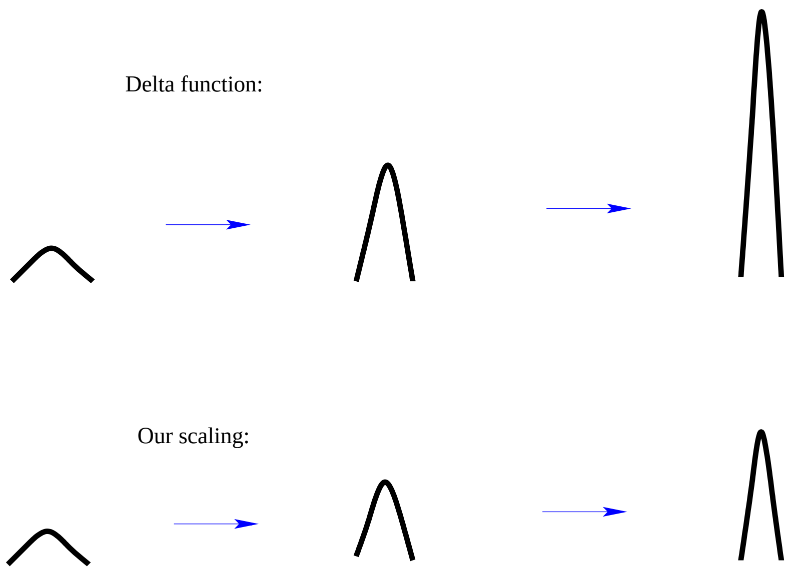
We want the body to shrink down to a worldline, γ, given by $x^i = z^i(t)$. In the usual point particle limit, the total charge would remain fixed as we shrink the body down. If we want the body to keep its shape exactly as it shrinks down, we would want to consider a one-parameter family of charge distributions, $\rho(\lambda; t, x^i)$ that behaves like

$$\rho(\lambda; t, x^i) = \lambda^{-3}\tilde{\rho}(t, \frac{x^i - z^i(t)}{\lambda})$$

with $\tilde{\rho}$ as smooth function of all of its arguments. Instead, we want the charge distribution to go to zero proportionally to the size of the body as $\lambda \to 0$. We also only demand that it retain its shape asymptotically as

$\lambda \rightarrow 0$. Thus, we require

$$\rho(\lambda; t, x^i) = \lambda^{-2} \tilde{\rho}(\lambda; t, \frac{x^i - z^i}{\lambda})$$



Precise Statement of How Our Charged Body Shrinks to Zero Size

We consider a one-parameter family $\{F_{\mu\nu}(\lambda), J^\mu(\lambda), T^M_{\mu\nu}(\lambda)\}$ of solutions to Maxwell's equations and conservation of total stress-energy such that

- $$J^\mu(\lambda, t, x^i) = \lambda^{-2}\tilde{J}^\mu(\lambda, t, [x^i - z^i(t)]/\lambda)$$

  and

  $$T^M_{\mu\nu}(\lambda, t, x^i) = \lambda^{-2}\tilde{T}_{\mu\nu}(\lambda, t, [x^i - z^i(t)]/\lambda)$$

  with $\tilde{J}^\mu$ and $\tilde{T}_{\mu\nu}$ smooth.

- We have $F_{\mu\nu} = F^{\text{ext}}_{\mu\nu} + F^{\text{ret}}_{\mu\nu}$, where $F^{\text{ret}}_{\mu\nu}$ is the retarded solution of Maxwell's equations with source $J^\mu(\lambda)$ and $F^{\text{ext}}_{\mu\nu}$ is a homogeneous solution of Maxwell's equation that is jointly smooth function of λ and the spacetime point.

We want to know (1) What are the possible worldlines $z^i(t)$ and (2) What are the perturbative corrections to $z^i(t)$ that arise from self-field and finite size effects?

Two Key Properties of $F_{\mu\nu}^{\text{ret}}$

In global inertial coordinates, we have

$$A_\mu^{\text{ret}}(\lambda, t, x^i) = \int \mathrm{d}^3x' \left[\frac{J_\mu(\lambda, t - |x^i - x'^i|, x'^j)}{|x^i - x'^i|}\right] .$$

Plugging in $J^\mu(\lambda, t, x^i) = \lambda^{-2}\tilde{J}^\mu(\lambda, t, [x^i - z^i(t)]/\lambda)$ with $\tilde{J}^\mu$ smooth, it is not difficult to show that

$$F_{\mu\nu}^{\text{ret}}(\lambda, t, x^i) = \lambda^{-1}\tilde{F}_{\mu\nu}(\lambda, t, [x^i - z^i(t)]/\lambda) ,$$

where $\tilde{F}$ is a smooth function of its arguments. (This general form is preserved under smooth coordinate transformations.) Thus, $F_{\mu\nu}^{\text{ret}}$ also behaves in an asymptotically self-similar manner near the worldline as $\lambda \to 0$.

Define $\beta = \lambda/r$, where $r = \sqrt{\sum[x^i - z^i(t)]^2}$. A much lengthier argument proves that

$$\lambda F^{\rm ret}_{\mu\nu} = \beta^2 \mathcal{F}_{\mu\nu}(t, \beta, r, \theta, \phi)$$

where $\mathcal{F}_{\mu\nu}$ is smooth in all of its arguments at $r = \beta = 0$. This means that we can approximate $F^{\rm ret}_{\mu\nu}$ by

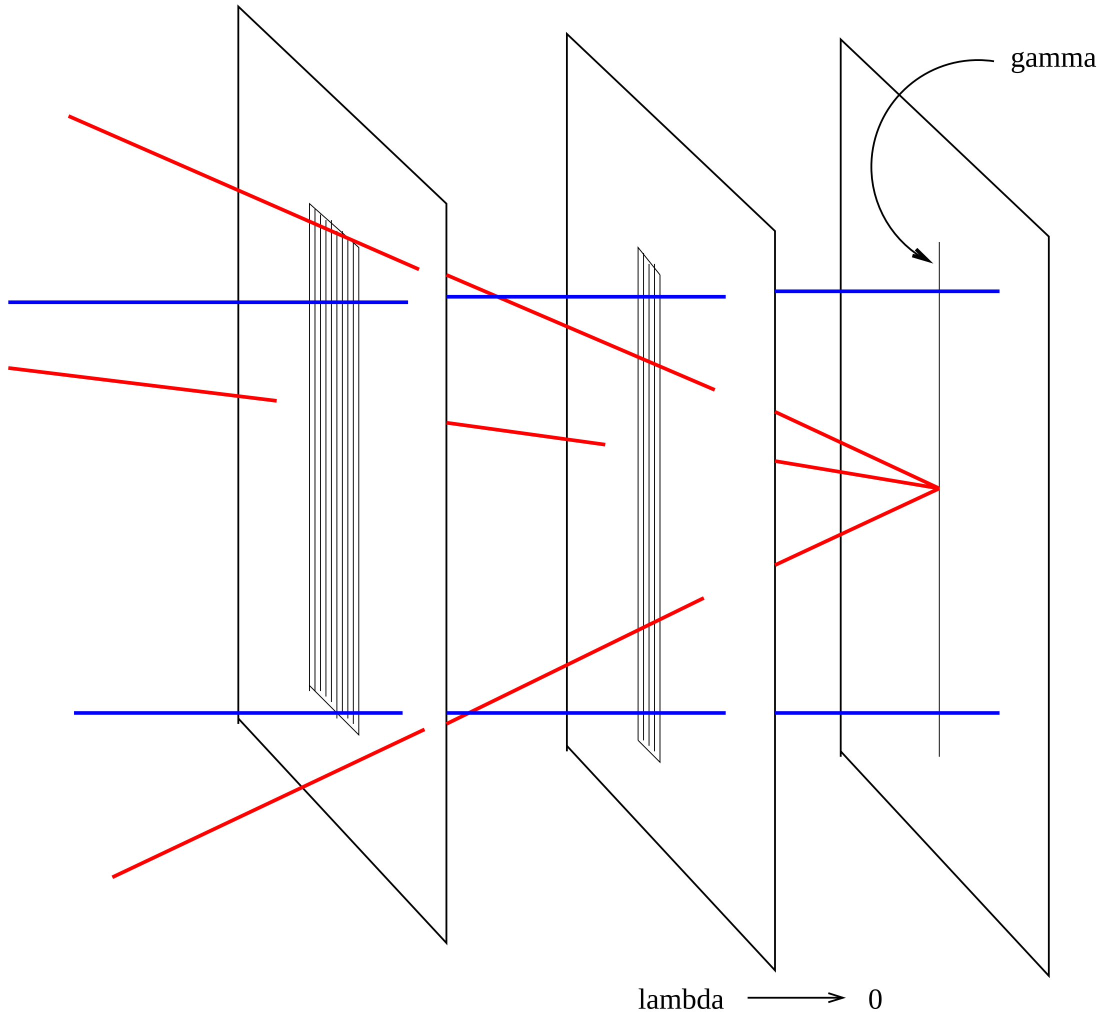
$$F^{\rm ret}_{\mu\nu}(t, r, \theta, \phi) = \frac{\lambda}{r^2} \sum_{n=0}^{N} \sum_{m=0}^{M} r^n \left(\frac{\lambda}{r}\right)^m (\mathcal{F}_{\mu\nu})_{nm}(t, \theta, \phi)$$

This gives a "far zone" expansion of $F^{\rm ret}_{\mu\nu}$, valid near $r = 0$. Alternatively, defining $\bar{r} = r/\lambda$, and $\bar{t} = (t - t_0)/\lambda$, we can rewrite this as a "near-zone" expansion

$$\lambda^{-1} F^{\text{ret}}_{\bar{\mu}\bar{\nu}}(\bar{t}, \bar{r}, \theta, \phi) = \sum_{n=0}^{N} \sum_{m=0}^{M} (\lambda \bar{r})^n \frac{1}{\bar{r}^{(m+2)}} (\mathcal{F}_{\mu\nu})_{nm}(t_0 + \lambda \bar{t}, \theta, \phi)$$

which is valid at *large* $\bar{r}$.

The "Near-Zone" and "Far-Zone" Limits



"Far Zone" Limit and Unperturbed Motion

Let $\lambda \to 0$ at fixed x^μ. Then $J^\mu(\lambda, t, x^i)$ can be expanded in a *distributional* series. We find that $J^{(0)\mu} \equiv \lim_{\lambda\to 0} J^\mu(\lambda) = 0$ and

$$J^{(1)\mu} \equiv \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} J^\mu(\lambda) = \mathcal{J}^\mu(t)\delta(x^i - z^i(t))$$

Conservation of J^μ then yields

$$J^{(1)\mu} = q u^\mu \delta(x^i - z^i(t)) \frac{d\tau}{dt} \ .$$

Similarly,

$$T^{M,(1)}_{\mu\nu} = \mathcal{T}^M_{\mu\nu}(t)\delta(x^i - z^i(t)) \ .$$

If we write,

$$T_{\mu\nu}^{EM} = T_{\mu\nu}^{\rm ext} + T_{\mu\nu}^{\rm cross} + T_{\mu\nu}^{\rm self} \ ,$$

then, remarkably, we find

$$T_{\mu\nu}^{\rm self,(1)} \equiv \lim_{\lambda\to 0} \frac{\partial}{\partial\lambda} T_{\mu\nu}^{\rm self}(\lambda) = \mathcal{T}_{\mu\nu}^{\rm self}(t)\delta(x^i - z^i(t)) \ .$$

Define $T_{\mu\nu} \equiv T_{\mu\nu}^{M} + T_{\mu\nu}^{\rm self}$. Conservation of total stress energy then yields

$$T_{\mu\nu}^{(1)}(t) = m u_\mu u_\nu \delta(x^i - z^i(t)) \frac{d\tau}{dt} \ ,$$

$$m u^\nu \nabla_\nu u_\mu = q u^\nu F_{\mu\nu}^{\rm ext}(\lambda = 0, t, z^i(t)) \ .$$

Thus, to first order in λ, the description of *any* body is is precisely that of a classical point charge/mass moving on a Lorentz force trajectory of the external field. Note that the electromagnetic self-energy of the body contributes to its mass.

“Near Zone” Limit and Perturbed Motion

As $\lambda \to 0$, the body shrinks down to the worldline γ defined by $x^i = z^i(t)$, which satisfies the Lorentz force equation. However, at any $\lambda > 0$, the body is of finite size, so in order to find the “correction” to γ at finite λ, we would need to have a notion of the “center of mass worldline” $\gamma(\lambda)$ of the body to represent its motion. This is highly nontrivial since “electromagnetic self-energy” must be included, but one does not want to include electromagnetic radiation that was emitted in the past. Fortunately, this can be done straightforwardly to the order needed to obtain first-order perturbed motion.

It is convenient to work in Fermi normal coordinates based on the worldline $\gamma(\lambda)$—so $z^i(\lambda, t) = 0$. To take the near-zone limit, we let $\lambda \to 0$ at fixed $\bar{x}^\mu$ rather than at fixed x^μ, where $\bar{t} \equiv (t - t_0)/\lambda$, $\bar{x}^i \equiv x^i/\lambda$. We also rescale the fields as follows:

$$\bar{g}_{\mu\nu} \equiv \lambda^{-2} g_{\mu\nu}$$

$$\bar{J}^\mu \equiv \lambda^3 J^\mu$$

$$\bar{T}^M_{\mu\nu} \equiv T^M_{\mu\nu}$$

$$\bar{F}_{\mu\nu} \equiv \lambda^{-1} F_{\mu\nu}$$

The rescaled fields then approach well defined, finite limits as $\lambda \to 0$. At $\lambda = 0$, the rescaled fields are stationary.

Center of Mass

Define

$$\bar{T}_{\bar{\mu}\bar{\nu}} \equiv \bar{T}^{M}_{\bar{\mu}\bar{\nu}} + \bar{T}^{\rm self}_{\bar{\mu}\bar{\nu}} \ ,$$

define the zeroth order near-zone mass by

$$m(t_0) \equiv \int \bar{T}^{(0)}_{00} d^3\bar{x}$$

and define the zeroth order near zone center of mass by

$$\bar{X}^i_{\rm CM}(t_0) = \frac{1}{m} \int \bar{T}^{(0)}_{00} \bar{x}^i d^3\bar{x} \ .$$

The perturbed motion is defined by the condition $\bar{X}^i_{\rm CM} = 0$.

Other Body Parameters

Spin tensor:

$$S^{0j} = -S^{j0} = \int \bar{T}^{(0)00} \bar{x}^j d^3\bar{x} = 0$$

$$S^{ij}(t_0) \equiv 2 \int \bar{T}^{(0)i}{}_0 \bar{x}^j d^3\bar{x}$$

Spin vector:

$$S_i = \frac{1}{2} \epsilon_{ijk} S^{jk}$$

Perturbed mass:

$$\delta m(t_0) \equiv \delta \int_\Sigma \bar{T}_{\mu 0} d\Sigma^\mu$$

Charge:

$$q \equiv \int \bar{J}^{(0)0} d^3\bar{x}$$

Perturbed charge:

$$\delta q = \delta \int_{\Sigma} \bar{J}^{\mu} d\Sigma_{\mu}$$

Electromagnetic dipole tensor:

$$Q^{\mu j}(t_0) \equiv \int \bar{J}^{(0)\mu} \bar{x}^{j}$$

Electric dipole moment:

$$p^{i} = Q^{0i}$$

Magnetic dipole moment:

$$\mu_i = -\frac{1}{2}\epsilon_{ijk}Q^{jk}$$

Derivation of Motion

Strategy: We write down the equations arising from conservation of total stress-energy and conservation of charge-current at 0th, 1st, and 2nd order in the near-zone expansion. We multiply these relations by various powers of $\bar{x}^i$ and integrate over space to systematically obtain all relationships holding for the body parameters defined above.

- At 0th order, we obtain various relationships, such as the antisymmetry of the spatial components of the spin and electromagnetic dipole tensors.

- At 1st order, we obtain other relationships including

$$\frac{d}{dt_0}m = 0 \ , \quad \frac{d}{dt_0}S_{ij} = -Q^{\mu}{}_{[i}F^{\mathrm{ext}}_{j]\mu} \ , \quad ma_i = qF^{\mathrm{ext}}_{0i} \ .$$

- At 2nd order, we obtain

$$\begin{aligned} m\delta a_i &= -(\delta m)a_i + (\delta q)F^{\mathrm{ext}}_{0i} + q\delta F^{\mathrm{ext}}_{0i} + \frac{2}{3}q^2\dot{a}_i + \\ & \quad + \frac{1}{2}Q^{\mu\nu}\partial_i F^{\mathrm{ext}}_{\mu\nu} + \frac{d}{dt_0}\left(a^j S_{ji} + 2Q^{j}{}_{[i}F^{\mathrm{ext}}_{0]j}\right) \\ \frac{d}{dt_0}\delta m &= \frac{1}{2}Q^{\mu\nu}\partial_0 F^{\mathrm{ext}}_{\nu\mu} - \frac{\partial}{\partial t_0}\left(Q^{\mu 0}F^{\mathrm{ext}}_{0\mu}\right) \end{aligned}$$

Note that there is no evolution equation for $Q^{\mu\nu}$.

Perturbed Equations of Motion in Covariant Form

Define

$$\delta\hat{m} \equiv \delta m - u_b u^c Q^{bd} F^{\text{ext}}_{cd} \ .$$

Then, we have

$$\delta[\hat{m} a_a] = \delta[q F^{\text{ext}}_{ab} u^b] + \left(g_a{}^b + u_a u^b\right) \left\{ \frac{2}{3} q^2 \frac{D}{d\tau} a_b + \frac{1}{2} Q^{cd} \nabla_b F^{\text{ext}}_{cd} + \frac{D}{d\tau} \left(a^c S_{cb} + 2 u^d Q^c{}_{[b} F^{\text{ext}}_{d]c} \right) \right\}$$

$$\frac{D}{d\tau} S_{ab} = -2 \left(g^a{}_c + u^a u_c\right) \left(g^b{}_d + u^b u_d\right) Q^e{}_{[c} F^{\text{ext}}_{d]e} - 2 a^c S_{c[a} u_{b]}$$

$$\frac{D}{d\tau} \delta\hat{m} = -\frac{1}{2} Q^{ab} \frac{D}{d\tau} F^{\text{ext}}_{ab} - 4 Q_a{}^b F^{\text{ext}}_{bc} a^{[c} u^{a]}$$

Non-Relativistic Form of Perturbed Force

$$\delta\vec{F} \equiv \delta(m\vec{a}) = \frac{2}{3}q^2\frac{d\vec{a}}{dt} + (\vec{p}\cdot\vec{E})\vec{a} + p_i\vec{\nabla}E^i + \mu_i\vec{\nabla}B^i$$
$$+ \frac{d}{dt}\left(\vec{S}\times\vec{a} + \vec{\mu}\times\vec{E} + \vec{p}\times\vec{B}\right)$$

The first term is the usual ALD force, which we have now derived as a perturbative correction to Lorentz force motion. The other terms are corrections due to the finite size of the body. The second term could be incorporated into the definition of δm. The remaining two terms on the first line are the standard electric and magnetic dipole forces. The terms on the second line are associated with "hidden momentum", i.e., the failure of p^i to equal mv^i.

The quantities on the right side of the perturbed equations of motion are to be evaluated on the zeroth order solution. Thus, the perturbed equations of motion remain second order in time and admit no “runaway” solutions.

Self-Consistent Motion

The equations we have just derived should provide a good description of the perturbative corrections to the Lorentz force, provided, of course, that they are locally small. However, even if the perturbative corrections are locally small, the effects they have on solutions will build up over time, and a perturbative description based on perturbing off of a single, fixed Lorentz force trajectory will be a poor approximation at late times. Can one improve upon the purely perturbative description given here so as to obtain a much better global in time description of motion? Note that going to any finite order in perturbation theory will not really help!

To improve the description of motion so that it remains accurate at late times, we would like to invent a *self consistent perturbative equation* that corrects the Lorentz force trajectory "as one goes along." In physics, people do this kind of thing all the time, usually without noticing. It should be OK to do this provided that the new equation satisfies the following properties: (1) It should have a well posed initial value formulation. (2) It should have the same number of degrees of freedom as the first order perturbative system, so that a correspondence can be made between initial data for the self-consistent perturbative equation and the first order perturbative system. (3) For corresponding initial data, the solutions to the self-consistent perturbative equation should be close to the corresponding solutions of the first order perturbative system over the time interval for which the first order perturbative description should be accurate.

I do not know of any reason why, for any given system, there need exist a self-consistent perturbative equation satisfying these criteria. In cases where a self-consistent perturbative equation satisfying these criteria does exist, I would not expect it to be unique.

The obvious thing to try is to combine the 0th and 1st order equations into a single equation that is then treated as though it were "exact." However, in the present case, we get (in the non-relativistic approximation and neglecting dipole terms):

$$m\vec{a} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) + \frac{2}{3}q^2\frac{d\vec{a}}{dt}$$

However, this is clearly unacceptable, since it changes the differential order of the system and introduces spurious solutions. A perfectly good alternative is to take this equation but replace $d\vec{a}/dt$ and $\vec{a}$ on the right side by $(q/m)[\vec{E} + \vec{v} \times \vec{B}]$. This "reduced order" version of the ALD equation should give an accurate description of the motion of a "point charge".

An application of Our Results to Freshman Physics

Release a magnetic dipole $\vec{\mu}$ from rest in the non-uniform field $\vec{B}$ of a magnet. For appropriate choice of alignment of the dipole, the force $\mu_i \vec{\nabla} B^i$ will be non-zero, so the dipole will start to move. Its kinetic energy will therefore increase. Normally, one accounts for this increase in kinetic energy by a compensating loss in "magnetic dipole interaction energy" $-\vec{\mu} \cdot \vec{B}$. However, this explanation cannot be correct: A magnetic field can "do no work" on a body, so the energy of the body itself (*not* counting any interaction energy with the external field) cannot change. Where does the kinetic energy of the body come from?

Answer: It comes from the rest mass of the body!