EXAMPLES OF PLANAR AUTONOMOUS SYSTEMS (use with pplane7)

Ex 1. (v. der Pol)

$$\begin{cases} x' &= y - Mx + x^3 \\ y' &= -x \end{cases}$$

Parameter values: $M = -0.5, -0.1, -0.05, 0, 0, 2$, window: $[-5, 5] \times [-5, 5]$.

The origin is a spiral sink for $M < 0$, spiral source for $M > 0$. The system undergoes a Hopf bifurcation, with a single stable periodic orbit for $M > 0$. When $M = 0$, the linearization at the origin is not hyperbolic (a center), but the origin is asymptotically stable: $V = x^2 + y^2$ is a (non-strict) Liapunov function, and LaSalle's invariance principle may be used.

Ex 2.(Lienard's theorem)

$$\begin{cases} x' &= y - f(x) \\ y' &= -x \end{cases}$$

(a) Let $f(x) = x(|x| - 1)(|x| - 2)(|x| - 3)$. This satisfies the hypothesis of Lienard's theorem (use the window $[-6, 6] \times [-6, 6]$.)

We observe three periodic orbits. Moving out from the origin: stable, unstable, stable. The origin itself is an unstable spiral. Contemplation of the x vs. t graphs is also interesting.

(b) Let $f(x) = x(|x| - 1)(|x| - 2)$. This is positive for $0 < x < 1$, so the hypothesis of Lienard's theorem do not hold. We still observe two periodic solutions, one repelling (inner), one attracting (outer). The origin is a stable spiral. The $x - t$ graph of a solution starting between the two periodic orbits illustrates the transition between two periodic behaviors.

Can you find a system of this type with a semi-stable periodic orbit? The naive guess $f(x) = x(|x| - 1)^2(|x| - 2)$ does not work (it seems to have a single, stable periodic orbit).

(c) One-parameter family: $f(x) = x(|x| - 1)(|x| - M)(|x| - 3)$. For $M = 2$ or $M = 1.8$, we have three periodic solutions. For $M = 1.6$, this is not so clear: a solution starting close to the origin seems to 'jump' to a large periodic orbit after winding around for a while, but (as remarked in class by Matt Dawson) this could simply be an artifact of the computation. (We won't know what really happens unless we can come up with a rigorous proof!)

Ex 3. Generalized predator-prey

$$\begin{cases} x' &= x(1-\frac{x}{4}) - \frac{2xy}{x+1} \\ y' &= -y + \frac{2xy}{x+1} \end{cases}$$

The prey population is $x(t)$, the predator $y(t)$. For the prey, logistic growth in the absence of predator is moderated by predator-prey encounters, which adjust the growth rate (downwards!) by an amount proportional to $y/(x+1)$ (hence, smaller adjustment with a lot of prey around- too many to catch!) The predator growth rate is increased by a similar amount.

We observe (with the window $[0,6]\times[0,6]$) a saddle point at the origin, and another on the $x-$axis at $(4,0)$. Most interestingly, an *unstable* spiral at $(1,0.77)$, and a *stable* periodic orbit, which is the $\omega-$ limit set of almost all solutions: the asymptotic state of the population is eternal oscillation in the populations of predator and prey, neither of which becomes totally extinct (cold comfort for the individual prey being chased, or the individual starving predator.) One can show (using the index) that the 'standard' (polynomial) Lotka-Volterra-type systems with limited growth cannot exhibit periodic solutions.