Old Evidence and New Explanation*

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To Keith Lehrer on the Occasion of his Sixtieth Birthday

Jeffrey has devised a probability revision method that increases the probability of hypothesis H when it is discovered that H implies previously known evidence E. A natural extension of Jeffrey's method likewise increases the probability of H when E has been established with sufficiently high probability and it is then discovered, quite apart from this, that H confers sufficiently higher probability on E than does its logical negation $\overline{H}$.

1. Introduction.

1.1 Old Explanation and New Evidence. If hypothesis H is known to imply the less-than-certain proposition E, the subsequent discovery that E is true ought to confirm (i.e., raise the probability of) H. There is a straightforward Bayesian account of such confirmation, for from $p(E|H) = 1 > p(E)$ it follows immediately that $p(H|E) > p(H)$.

Indeed, if H is merely positively relevant to E under the prior p and new evidence prompts a revision of p to p^* by probability kinematics (Jeffrey 1983, 1988) on the partition $\{E, \overline{E}\}$, with $p^*(E) > p(E)$, then $p^*(H) > p(H)$. For

$$\begin{aligned} p^*(H) - p(H) &= (p^*(E) - p(E))p(H|E) + (p^*(\overline{E}) - p(\overline{E}))p(H|\overline{E}) \\ &= (p^*(E) - p(E))(p(H|E) - p(H|\overline{E})) > 0, \end{aligned}$$

the first factor being positive by assumption and the second by the positive relevance of H to E (hence, of E to H). Confirmation here is simply a matter of the symmetry of positive relevance.

More generally, one may prove the following theorem.

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Theorem 1. *Let $\{E_i\}$ be a finite or countably infinite partition of the set of possible states of the world, with $p(H) > 0$ and $p(E_i) > 0$ for all i. Let I be a nonempty, proper subset of the set of indices i, with $p(E_i|H) > p(E_i)$ if $i \in I$ and $p(E_i|H) \leq p(E_i)$ if $i \notin I$. Let p^* come from p by probability kinematics on the partition $\{E_i\}$. Suppose that $p^*(E_i) \geq p(E_i)$ if $i \in I$, with $p^*(E_i) > p(E_i)$ for some $i \in I$, and that $p^*(E_i) \leq p(E_i)$ if $i \notin I$. Then $p^*(H) > p(H)$.*

Proof: Clearly,

$$\begin{aligned} p^*(H) - p(H) &= \sum_i (p^*(E_i) - p(E_i))p(H|E_i) \\ &= \sum_{i \in I} (p^*(E_i) - p(E_i))p(H|E_i) + \sum_{i \notin I} (p^*(E_i) - p(E_i))p(H|E_i), \end{aligned}$$

so it suffices to show that

$$\sum_{i \in I} (p^*(E_i) - p(E_i))p(H|E_i) > \sum_{i \notin I} (p(E_i) - p^*(E_i))p(H|E_i). \quad (1)$$

By the symmetry of positive relevance it follows that $p(H|E_i) > p(H)$ if $i \in I$, and $p(H|E_i) \leq p(H)$ if $i \notin I$. Hence,

$$\begin{aligned} &\sum_{i \in I} (p^*(E_i) - p(E_i))p(H|E_i) > p(H) \sum_{i \in I} (p^*(E_i) - p(E_i)) \\ &= p(H) \sum_{i \notin I} (p(E_i) - p^*(E_i)) \geq \sum_{i \notin I} (p(E_i) - p^*(E_i))p(H|E_i), \end{aligned}$$

which establishes (1). □

1.2 Old Evidence and New Explanation. Suppose that we first attain certainty regarding E and subsequently discover, quite apart from this certainty, that H implies E.[1] Just as it does when explanation precedes observation, this explanation of the previously known fact E by the hypothesis H ought to increase the probability of H. As Clark Glymour (1980) has noted, however, such an increase cannot come about by conditioning on E, for $p(H|E) = p(H)$ if, as would here be the case for the prior p, $p(E) = 1$. Glymour has termed this dilemma *the problem of old evidence,*[2] and views it as a major challenge to Bayesian confirmation theory.

One proposed solution to this problem, due to Daniel Garber (1983),

1. The canonical example here is Einstein's explanation of the previously observed advance in the perihelion of Mercury in terms of the general theory of relativity. See, e.g., Weinberg 1992, 94.

2. Glymour actually identified more than one problem of old evidence. The problem we consider here is what Garber (1983), influenced by Skyrms, calls the *historical problem of old evidence.* Jeffrey (1991, 1995), on the other hand, calls it the *problem of new explanation.*

extends the algebra on which probabilities are defined to include the proposition $H \vdash E$ that H implies E, and then conditions on $H \vdash E$. Under certain conditions, $p(H|H \vdash E) > p(H)$. Richard Jeffrey (1991, 1995) has proposed a solution that retains the original algebra, but revises probabilities by an entirely new method called *reparation.* A key feature of Jeffrey's solution is the imaginative reconstruction of a probability distribution (the *ur-distribution*) that predates both our certainty regarding E and our discovery that H implies E.

In this paper Jeffrey's method is extended to give an account of confirmation in the case of old probable evidence and new probabilistic explanation. The extended method increases the probability of H when E has been established with sufficiently high probability and it is then discovered, quite apart from this, that H confers sufficiently higher probability on E than does its logical negation $\overline{H}$.

In what follows, I describe Jeffrey's method in §2 and extend it to the case of probable evidence and probabilistic explanation in §3, concluding in §3.5 with a discussion of the case in which the evidence bearing on H comprises an arbitrary finite partition of the set of possible states of the world. As in Jeffrey's analysis, the proposed revision of a prior based on new probabilistic explanation can be derived from either of two appealing heuristic principles, the *uniformity principle* or the *commutativity principle.* The first of these dictates (in one of its several equivalent formulations) that explanation-based probability revisions should preserve certain ratios of new-to-old odds. The second specifies that revisions based on observation and explanation should result in the same outcome, regardless of the order in which they are applied.

2. Jeffrey's Solution. Suppose that p is a probability distribution on the algebra of propositions generated by E and H and that $p(E) = 1$, reflecting our certainty about the truth of E:

$$\begin{array}{lcccc} & HE & H\overline{E} & \overline{H}E & \overline{H}\overline{E} \\ p\text{:} & \alpha & 0 & 1-\alpha & 0. \end{array} \tag{2}$$

We then discover, *quite apart from this certainty,* that H implies E. Jeffrey's strategy for revising p in the light of this discovery accordingly has us imagine a distribution p_0 (the *ur-distribution*) that predates both our certainty about E and our discovery that H implies E:

$$p_0\text{:} \quad a \quad b \quad c \quad d. \tag{3}$$

It is assumed that a, b, c, and d are positive, and that p has come from p_0 by conditioning on E, so that

$$\alpha = a/a + c \text{ and } 1 - \alpha = c/a + c. \tag{4}$$

If p_0 were our prior and we discovered that H implies E, but nothing new about the relation between $\overline{H}$ and E, or about the probability of H, it would be reasonable to revise p_0 to a distribution p_1 satisfying

$$p_1(E|H) = 1 \tag{5}$$
$$p_1(E|\overline{H}) = p_0(E|\overline{H}),^3 \text{ and} \tag{6}$$
$$p_1(H) = p_0(H). \tag{7}$$

Jeffrey (1991) expresses conditions (5)–(7) in the alternative, equivalent[4] form

$$p_1(E|H)/p_1(\overline{E}|H) = \infty, \tag{8}$$
$$p_1(E|\overline{H})/p_1(\overline{E}|\overline{H}) = p_0(E|\overline{H})/p_0(\overline{E}|\overline{H}), \text{ and} \tag{9}$$
$$p_1(H|E)/p_1(\overline{H}|E) = p_0(H|E)/p_0(\overline{H}|E)p_0(E|H). \tag{10}$$

There is exactly one distribution p_1 satisfying (5), (6), and (7) or, equivalently, (8), (9), and (10):

$$p_1\colon \quad a + b \quad 0 \quad c \quad d. \tag{11}$$

Jeffrey now reasons that the revision of p (call it p^*) prompted by the discovery that H implies E should bear the same relation to p as

3. That one must judge, case-by-case, whether (6) is appropriately assumed was first emphasized by Diaconis (Jeffrey 1991, 106). If, for example, $\overline{H}$ included alternative hypotheses which also implied or conferred high probability on E, this assumption might not be appropriate.

4. The proof that the conjunction of conditions (5)–(7) is equivalent to the conjunction of conditions (8)–(10) goes as follows: Clearly, (5) implies (8), and (6) implies (9). Furthermore, by the odds form of Bayes' rule,

$$\begin{aligned}\frac{p_1(H|E)}{p_1(\overline{H}|E)} &= \frac{p_1(H)}{p_1(\overline{H})}\cdot\frac{p_1(E|H)}{p_1(E|\overline{H})}\\ &= \frac{p_0(H)}{p_0(\overline{H})}\cdot\frac{1}{p_0(E|\overline{H})} \text{ by (7), (5), and (6)}\\ &= \frac{p_0(H|E)}{p_0(\overline{H}|E)}\cdot\frac{1}{p_0(E|H)},\end{aligned}$$

which establishes (10). Conversely, it is clear that (8) implies (5), and (9) implies (6). Finally,

$$\begin{aligned}\frac{p_1(H)}{p_1(\overline{H})} &= \frac{p_1(H|E)}{p_1(\overline{H}|E)}\cdot\frac{p_1(E|\overline{H})}{p_1(E|H)}\\ &= \frac{p_0(H|E)}{p_0(\overline{H}|E)}\cdot\frac{p_0(E|\overline{H})}{p_0(E|H)} \text{ by (10), (6), and (5)}\\ &= \frac{p_0(H)}{p_0(\overline{H})},\end{aligned}$$

which yields 7.

the explanation-based revision p_1 of p_0 does to p_0, the latter as captured by the conditions (8)–(10). Thus p^* should satisfy the conditions

$$p^*(E|H)/p^*(\overline{E}|H) = \infty, \quad (12)$$
$$p^*(E|\overline{H})/p^*(\overline{E}|\overline{H}) = p(E|\overline{H})/p(\overline{E}|\overline{H}), \text{ and} \quad (13)$$
$$p^*(H|E)/p^*(\overline{H}|E) = p(H|E)/p(\overline{H}|E)p_0(E|H). \quad (14)$$

As detailed in §3 below, conditions (12)–(14) represent a special case of a uniformity principle for certain conditional Bayes factors (i.e., ratios of new-to-old conditional odds) associated with explanation-based probability revisions. In any case, there is exactly one distribution p^* satisfying (12), (13), and (14):

$$p^*: \quad \frac{a+b}{a+b+c} \quad 0 \quad \frac{c}{a+b+c} \quad 0. \quad (15)$$

To wrap things up, we need only check that, as desired,

$$p^*(H) = \frac{a+b}{a+b+c} > \frac{a}{a+c} = p(H). \quad (16)$$

Jeffrey observes that p^* can also be derived from a *principle of commutativity,* to the effect that probability revisions based on observation and explanation should result in the same outcome, regardless of the order in which they are applied. Having revised p_0 to p_1 based on the (implicational) explanation of E by H, we would, upon observing E, revise p_1 to $p_1(\cdot|E)$, the result of conditioning p_1 on E. The commutativity principle would therefore dictate $p_1(\cdot|E)$ as the proper revision of p based on the discovery that H implies E. Happily, $p_1(\cdot|E) = p^*$, as an immediate consequence of (11) and (15). We shall see in the next section the same concordance of uniformity and commutativity principles in the solution of the problem of old probable evidence and new probabilistic explanation.

3. Probable Evidence and Probabilistic Explanation.

3.1 Preliminaries. If q_1 and q_2 are probability distributions, with q_2 being a revision of q_1, and A_1, A_2, and B are propositions, the *Bayes factor* $\beta_{q_2,q_1}(A_1\!:\!A_2)$ is simply the ratio

$$\beta_{q_2,q_1}(A_1\!:\!A_2) := \frac{q_2(A_1)}{q_2(A_2)} \bigg/ \frac{q_1(A_1)}{q_1(A_2)} \quad (17)$$

of new-to-old odds, and the *conditional Bayes factor* $\beta_{q_2,q_1}(A_1\!:\!A_2|B)$ the ratio

$$\beta_{q_2,q_1}(A_1:A_2|B) := \frac{q_2(A_1|B)}{q_2(A_2|B)} \bigg/ \frac{q_1(A_1|B)}{q_1(A_2|B)} \tag{18}$$

of new-to-old conditional odds.

When $q_1 = q$, $q_2 = q(\cdot|E)$, $A_1 = H$, and $A_2 = \overline{H}$, the Bayes factor $\beta_{q_2,q_1}(A_1:A_2)$ is denoted by $\lambda_q(H,E)$ and called the *likelihood ratio* of hypothesis H on evidence E. By the odds form of Bayes' rule,

$$\frac{q(H|E)}{q(\overline{H}|E)} = \frac{q(H)}{q(\overline{H})}\frac{q(E|H)}{q(E|\overline{H})}, \tag{19}$$

we have the simple formula

$$\lambda_q(H,E) = q(E|H)/q(E|\overline{H}). \tag{20}$$

Bayes factors and likelihood ratios play a central role in the formulation and solution of the problem of old probable evidence and new probabilistic explanation.

3.2 Reparation generalized. Suppose that p is a probability distribution on the algebra generated by E and H:

$$\begin{array}{ccccc} & HE & H\overline{E} & \overline{H}E & \overline{H}\overline{E} \\ p\colon & \alpha & \beta & \gamma & \delta. \end{array} \tag{21}$$

Certain observations have led to our assigning $p(E)$ a high value, in a sense that will later be made more precise. We subsequently discover, *quite apart from the observations underlying p,* theoretical considerations that, *taken alone,* indicate that the truth of H would confer probability v on E, and its falsity would confer probability u on E. Taken alone, these theoretical considerations would not have tended to alter whatever probability was ascribed to H prior to the assessment of p. How should p be revised in the light of this theoretical discovery? We describe below a generalization of Jeffrey's method that raises the probability of H when $p(E)$ and the ratio v/u are sufficiently large.

As in Jeffrey's approach, we resurrect a notional ur-distribution p_0 predating both observation and explanation:

$$p_0\colon \quad a \quad b \quad c \quad d. \tag{22}$$

The conceit is that p has come from p_0 by probability kinematics on the partition $\{E, \overline{E}\}$. Thus

$$\alpha/\gamma = a/c \quad \text{and} \quad \beta/\delta = b/d. \tag{23}$$

In logical effect (though not in historical reality) it is the conceptual state captured by p_0 in which we make the aforementioned theoretical discovery. This discovery would of course lead us to revise p_0 to a distribution p_1 satisfying

$$p_1(E|H) = v \tag{24}$$
$$p_1(E|\overline{H}) = u, \text{ and} \tag{25}$$
$$p_1(H) = p_0(H). \tag{26}$$

There is just one such distribution:

$$p_1\text{:} \quad v(a+b) \quad (1-v)(a+b) \quad u(c+d) \quad (1-u)(c+d). \tag{27}$$

We now revise p to a distribution p^* in such a way that p^* bears the same relation to p as p_1 does to p_0. The following theorem delineates three equivalent formulations of this relation.

Theorem 2. *If* $\mathcal{A} := \{HE, H\overline{E}, \overline{H}E, \overline{H}\overline{E}\}$, *the conditions*

$$\beta_{p^*,p}(A_1{:}A_2) = \beta_{p_1,p_0}(A_1{:}A_2),\ \forall A_1, A_2 \in \mathcal{A}, \tag{28}$$
$$p^*(A) \propto p(A)p_1(A)/p_0(A),\ \forall A \in \mathcal{A}, \text{ and} \tag{29}$$
$$\begin{aligned}&(i)\ \beta_{p^*,p}(H{:}\overline{H}|E) = \beta_{p_1,p_0}(H{:}\overline{H}|E)\\ &(ii)\ \beta_{p^*,p}(H{:}\overline{H}|\overline{E}) = \beta_{p_1,p_0}(H{:}\overline{H}|\overline{E}),\\ &(iii)\ \beta_{p^*,p}(E{:}\overline{E}|H) = \beta_{p_1,p_0}(E{:}\overline{E}|H), \text{ and}\\ &(iv)\ \beta_{p^*,p}(E{:}\overline{E}|\overline{H}) = \beta_{p_1,p_0}(E{:}\overline{E}|\overline{H})\end{aligned} \tag{30}$$

are equivalent, where the symbol $\propto$ *in (29) denotes proportionality.*
Proof.[5] □

We shall call the common principle underlying the equivalent con-

5. From (29) one gets, for all $A \in \mathcal{A}$, the exact formula $p^*(A) = p(A)p_1(A)/sp_0(A)$, where $s = p(HE)p_1(HE)/p_0(HE) + p(H\overline{E})p_1(H\overline{E})/p_0(H\overline{E}) + p(\overline{H}E)/p_1(\overline{H}E)/p_0(\overline{H}E) + p(\overline{H}\overline{E})$ $p_1(\overline{H}\overline{E})/p_0(\overline{H}\overline{E})$. So it is clear that (29) implies (28). Conversely, given (28), choose and fix some $\mathcal{B} \in \mathcal{A}$. Then (28) yields, for all $A \in \mathcal{A}$, that

$$p^*(A) = p(A)\frac{p_1(A)}{p_0(A)} \times \frac{p_0(B)p^*(B)}{p(B)p_1(B)},$$

which yields (29), the constant of proportionality being $p_0(B)p^*(B)/p(B)p_1(B)$.

It is straightforward to verify that (30)(i) is equivalent to the case $A_1 = HE$ and $A_2 = \overline{H}E$ of (28). Similarly, (30)(ii), (30)(iii), and (30)(iv) are equivalent, respectively, to the cases $A_1 = H\overline{E}$ and $A_2 = \overline{H}\overline{E}$, $A_1 = HE$ and $A_2 = H\overline{E}$, and $A_1 = \overline{H}E$ and $A_2 = \overline{H}\overline{E}$ of (28). It is simply a matter of tedious algebra to check that these four cases of (28) imply that (28) holds for all $A_1, A_2 \in \mathcal{A}$. Indeed, any three of these four cases of (28), hence any three of the conditions (30)(i)–(30)(iv), imply that (28) holds for all A_1, $A_2 \in \mathcal{A}$.

Condition (28) is a uniformity condition on certain Bayes factors associated with explanation-based probability revisions, and condition (30) a uniformity condition on certain conditional Bayes factors. Clearly, (30) generalizes Jeffrey's conditions (12)–(14), and was, indeed, suggested by these conditions. Condition (29), which posits the proportionality of the "relevance quotients" $p^*(A)/p(A)$ and $p_1(A)/p_0(A)$, which originated in Jeffrey (1995).

ditions (28), (29), and (30) the *uniformity principle* for explanation-based probability revisions. There is a unique revision p^* of p satisfying the uniformity principle:

$$p^*: \frac{v(a+b)\alpha}{as} \quad \frac{(1-v)(a+b)\beta}{bs} \quad \frac{u(c+d)\gamma}{cs} \quad \frac{(1-u)(c+d)\delta}{ds}, \tag{31}$$

where $s = v(a+b)\,\alpha/a + (1-v)(a+b)\beta/b + u(c+d)\gamma/c + (1-u)(c+d)\delta/d$. Note that (31) reduces to Jeffrey's (15) when $v = 1$, $u = c/c + d$, $\alpha = a/a + c$, $\gamma = c/a + c$, and $\beta = \delta = 0$, i.e., in the case of certain evidence and implicational explanation.

The following theorem states a condition sufficient to ensure that the desired inequality $p^*(H) > p(H)$ obtains for sufficiently large values of $p(E)$.

Theorem 3. *If $\lambda_{p_1}(H, E) > \lambda_{p_0}(H, E)$, then $p^*(H) > p(H)$ for sufficiently large values of $p(E)$.*

Proof. Since $p^*(H) > p(H)$ if and only if $p^*(H)/p^*(\overline{H}) > p(H)/p(\overline{H})$, it suffices to show that

$$\lim_{p(E)\to 1} \frac{p^*(H)}{p^*(\overline{H})} - \frac{p(H)}{p(\overline{H})} > 0. \tag{32}$$

But

$$\lim_{p(E)\to 1} \frac{p^*(H)}{p^*(\overline{H})} = \frac{\lambda_{p_1}(H,E)}{\lambda_{p_0}(H,E)} \times \frac{p_0(H|E)}{p_0(\overline{H}|E)} \text{[6]} \tag{33}$$

and

6. Since $p_1(H) = p_0(H)$ by (26), we have

$$\lambda_{p_1}(H,E)/\lambda_{p_0}(H,E) = p_1(EH)p_0(E\overline{H})/p_0(EH)p_1(E\overline{H}). \tag{i}$$

By (29),

$$\frac{p^*(H)}{p^*(\overline{H})} = \frac{p^*(EH) + p^*(\overline{E}H)}{p^*(E\overline{H}) + p^*(\overline{E}\overline{H})} = \frac{\dfrac{p(E)p(H|E)p_1(EH)}{p_0(EH)} + \dfrac{p(\overline{E}H)p_1(\overline{E}H)}{p_0(\overline{E}H)}}{\dfrac{p(E)p(\overline{H}|E)p_1(E\overline{H})}{p_0(E\overline{H})} + \dfrac{p(\overline{E}\overline{H})p_1(\overline{E}\overline{H})}{p_0(\overline{E}\overline{H})}} \tag{ii}$$

Since p comes from p_0 by probability kinematics on $\{E, \overline{E}\}$, $p(H|E) = p_0(H|E)$ and so $p(\overline{H}|E) = p_0(\overline{H}|E)$. Along with (ii), this implies that

$$\begin{aligned}\lim_{p(E)\to 1} \frac{p^*(H)}{p^*(\overline{H})} &= \frac{p_1(EH)p_0(E\overline{H})p_0(H|E)}{p_0(EH)p_1(E\overline{H})p_0(\overline{H}|E)} \\ &= \frac{\lambda_{p_1}(H,E)}{\lambda_{p_0}(H,E)} \times \frac{p_0(H|E)}{p_0(\overline{H}|E)},\end{aligned} \tag{iii}$$

by (i).

$$\lim_{p(E)\to 1} \frac{p(H)}{p(\overline{H})} = \frac{p_0(H|E)}{p_0(\overline{H}|E)}.^{7} \tag{34}$$

Since $\lambda_{p_1}(H, E) > \lambda_{p_0}(H, E)$, (33) and (34) imply (32). □

Thus we see that generalized reparation increases the probability of H when E has been established with sufficiently high probability and it is then discovered, quite apart from this, that H confers sufficiently higher probability on E then does its logical negation $\overline{H}$, specifically, when $\lambda_{p_1}(H, E) = v/u > \lambda_{p_0}(H, E)$. When $p(E) = 1$ (the case of certain evidence and probabilistic explanation), the condition $\lambda_{p_1}(H, E) > \lambda_{p_0}(H, E)$ is in fact necessary and sufficient to ensure that $p^*(H) > p(H)$.[8] In the case of certain evidence and implicational explanation, this condition always holds[9] and so, as Jeffrey noted, H is always confirmed.

7. Since p comes from p_0 by probability kinematics on $\{E, \overline{E}\}$,

$$\lim_{p(E)\to 1} \frac{p(H)}{p(\overline{H})} = \lim_{p(E)\to 1} \frac{p(E)p_0(H|E) + p(\overline{E})p_0(H|\overline{E})}{p(E)p_0(\overline{H}|E) + p(\overline{E})p_0(\overline{H}|\overline{E})} = \frac{p_0(H|E)}{p_0(\overline{H}|E)},$$

as desired. We have simply shown here the intuitively obvious fact that the limiting case of probability kinematics, as $p(E) \to 1$, is ordinary conditioning on E.

8. By note 6, formula (i),

$$\begin{aligned}\lambda_{p_1}(H,E) > \lambda_{p_0}(H,E) &\Leftrightarrow p_1(EH)p_0(E\overline{H}) > p_0(EH)p_1(E\overline{H}) \\ &\Leftrightarrow \frac{p_1(H|E)}{p_1(\overline{H}|E)} > \frac{p_0(H|E)}{p_0(\overline{H}|E)} \\ &\Leftrightarrow p_1(H|E) > p_0(H|E) = p(H).\end{aligned} \tag{i}$$

By (27) the distribution p_1 is given by

$$\begin{array}{lcccc} & HE & H\overline{E} & \overline{H}E & \overline{H}\overline{E} \\ p_1: & v(a+b) & (1-v)(a+b) & u(c+d) & (1-u)(c+d)\end{array} \tag{ii}$$

In this case, where $p(E) = 1$, formula (31) for p^* simplifies to

$$p^*: \quad \frac{v(a+b)}{v(a+b)+u(c+d)} \quad 0 \quad \frac{u(c+d)}{v(a+b)+u(c+d)} \quad 0. \tag{iii}$$

From (ii) and (iii) it is obvious that $p^*(H) = p_1(H|E)$. With (i) this establishes the asserted equivalence. Note that what we have just proved is that p^* may be derived from a commutativity principle as well as from the uniformity principle when $p(E) = 1$. We shall establish this result in the general case in §3.3.

9. For in this case,

$$\lambda_{p_1}(H,E) = \frac{p_1(E|H)}{p_1(E|\overline{H})} = \frac{1}{p_0(E|\overline{H})} > \frac{p_0(E|H)}{p_0(E|\overline{H})} = \lambda_{p_0}(H,E),$$

since $p_0(E|H) = a/a + b < 1$.

3.3. Generalized Reparation and Commutativity. The explanation-based revision p^* of p, given by (31), was derived from the uniformity principle. To show that p^* may also be derived from the commutativity principle, we need to show that p^* comes from p_1 by probability kinematics on the partition $\{E, \overline{E}\}$, just as p comes from p_0 by probability kinematics on that partition. This is established in the following theorem.

Theorem 4. *The distribution p^* comes from p_1 by probability kinematics on $\{E, \overline{E}\}$, with*

$$\beta_{p^*,p_1}(E:\overline{E}) = \beta_{p,p_0}(E:\overline{E}). \tag{35}$$

Proof. Since p comes from p_0 by probability kinematics on $\{E, \overline{E}\}$, we have

$$\frac{p(HE)}{p(\overline{H}E)} = \frac{p_0(HE)}{p_0(\overline{H}E)}, \text{ and} \tag{36}$$

$$\frac{p(H\overline{E})}{p(\overline{H}\overline{E})} = \frac{p_0(H\overline{E})}{p_0(\overline{H}\overline{E})}. \tag{37}$$

To show that p^* comes from p_1 by probability kinematics on $\{E, \overline{E}\}$, it suffices to show that

$$\frac{p^*(HE)}{p^*(\overline{H}E)} = \frac{p_1(HE)}{p_1(\overline{H}E)}, \text{ and} \tag{38}$$

$$\frac{p^*(H\overline{E})}{p^*(\overline{H}\overline{E})} = \frac{p_1(H\overline{E})}{p_1(\overline{H}\overline{E})}. \tag{39}$$

Formula (38) follows from (28) with $A_1 = HE$ and $A_2 = \overline{H}E$, along with (36). Formula (39) follows from (28) with $A_1 = H\overline{E}$ and $A_2 = \overline{H}\overline{E}$, along with (37).

To prove (35), note first that by (29), (36), and (37),

$$\frac{p^*(E)}{p^*(\overline{E})} = \frac{p^*(HE) + p^*(\overline{H}E)}{p^*(H\overline{E}) + p^*(\overline{H}\overline{E})} = \frac{\dfrac{p(HE)p_1(HE)}{p_0(HE)} + \dfrac{p(\overline{H}E)p_1(\overline{H}E)}{p_0(\overline{H}E)}}{\dfrac{p(H\overline{E})p_1(H\overline{E})}{p_0(H\overline{E})} + \dfrac{p(\overline{H}\overline{E})p_1(\overline{H}\overline{E})}{p_0(\overline{H}\overline{E})}} = \frac{\dfrac{p(HE)}{p_0(HE)}}{\dfrac{p(H\overline{E})}{p_0(H\overline{E})}} \times \frac{p_1(E)}{p_1(\overline{E})}. \tag{40}$$

Also,

$$\frac{p(E)}{p(\overline{E})} = \frac{\dfrac{p(HE)p_0(HE)}{p_0(HE)} + \dfrac{p(\overline{H}E)p_0(\overline{H}E)}{p_0(\overline{H}E)}}{\dfrac{p(H\overline{E})p_0(H\overline{E})}{p_0(H\overline{E})} + \dfrac{p(\overline{H}\overline{E})p_0(\overline{H}\overline{E})}{p_0(\overline{H}\overline{E})}} = \frac{\dfrac{p(HE)}{p_0(HE)}}{\dfrac{p(H\overline{E})}{p_0(H\overline{E})}} \times \frac{p_0(E)}{p_0(\overline{E})}. \quad (41)$$

Formula (35) follows from (17), (40), and (41). □

Note that by (35) the revision of p_1 to p^* is effected by the "same" kinematical transformation that effects the revision of p_0 to p, not in the sense of assigning the same new probability to E, but rather in the sense of revising the old odds on E by the same factor.

3.4 Example. The following numerical example may clarify the ideas in §3.1–3.3. Suppose that our current distribution p on the algebra generated by H and E is given by

$$p: \quad \begin{matrix} HE & H\overline{E} & \overline{H}E & \overline{H}\overline{E} \\ \frac{4}{10} & \frac{1}{10} & \frac{4}{10} & \frac{1}{10}. \end{matrix} \quad (42)$$

We subsequently discover, quite apart from the observations underlying p, theoretical considerations that, taken alone, indicate that H would confer probability $v = .80$ on E and $\overline{H}$ would confer probability $u = .40$ on E. Taken alone, these theoretical considerations would not have tended to alter whatever probability was ascribed to H prior to the assessment of p. Imagine that p has come from an ur-distribution p_0 by probability kinematics on $\{E, \overline{E}\}$, with the p-odds on E quadrupling the p_0-odds on E, whence

$$p_0: \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}. \quad (43)$$

In logical effect, it is the conceptual state captured by p_0 in which we make the aforementioned theoretical discovery. This discovery would prompt a revision of p_0 to p_1, where

$$p_1: \quad \frac{4}{10} \quad \frac{1}{10} \quad \frac{2}{10} \quad \frac{3}{10}. \quad (44)$$

The revision p^* of p prompted by that theoretical discovery can be derived either from the uniformity principle or from the commutativity principle. The uniformity principle dictates that $p^*(HE) \propto (.40)(.40)/(.25)$, $p^*(H\overline{E}) \propto (.10)(.10)/(.25)$, $p^*(\overline{H}E) \propto (.40)(.20)/(.25)$, and $p^*(\overline{H}\overline{E}) \propto (.10)(.30)/(.25)$. Thus,

$$p^*: \quad \frac{16}{28} \quad \frac{1}{28} \quad \frac{8}{28} \quad \frac{3}{28}. \tag{45}$$

In this case, $p^*(H) = {}^{17}/_{28} > {}^{1}/_{2} = p(H)$. To derive p^* by the commutativity principle, we revise p_1 by probability kinematics on $\{E, \overline{E}\}$, choosing $p^*(E)$ so that the p^*-odds on E quadruple the p_1-odds on E, i.e., so that $p^*(E)/p^*(\overline{E}) = (4)(.60)/(.40) = 6$. This yields $p^*(E) = {}^{6}/_{7}$ and $p^*(\overline{E}) = {}^{1}/_{7}$. Revising p_1 to p^* by probability kinematics on $\{E, \overline{E}\}$ using these new probabilities of E and $\overline{E}$ again yields (45).[10]

3.5 Finer Evidentiary Partitions. In this section we generalize Theorem 3 to an old evidence/new explanation counterpart of Theorem 1. Here, however, we assume a finite partition $\{E_1, \ldots, E_n\}$ of the set of possible states of the world. Our current probability p is defined on the algebra of events generated by the hypothesis H and all of the evidentiary events E_i. There is a nonempty proper subset J of $\{1, \ldots, n\}$ with

10. At first glance, this example may look strange. Isn't generalized reparation supposed to be based on the discovery of a new, "stronger" relationship between H and E? Here, however, we have $v = .80 = p(E|H)$. Moreover, at the end of the revision exercise, we have $p^*(E|H) = 16/17 > .80$, which appears to contradict the discovery that H confers probability $v = .80$ on E.

Both these objections are based on a misunderstanding of new probabilistic explanation, which here amounts to the discovery, *based solely on theoretical grounds and quite apart from the observations underlying p*, that H confers probability $v = .80$ on E and $\overline{H}$ confers probability $u = .40$ on E. Since p_0, not p, captures the conceptual state in which this discovery is made, the judgement that a new, stronger relationship between H and E has been discovered is not based on a comparison of $p(E|H)$ with v. Nor is it based simply on a comparison of $p_0(E|H)$ with v. As shown in Theorem 3, it is the condition $\lambda_{p_1}(H, E) = v/u > \lambda_{p_0}(H, E)$ that captures the notion of a new, stronger relationship between H and E and sets the stage for the inequality $p^*(H) > p(H)$ to obtain for sufficiently large values of $p(E)$. It is instructive to redo this example with $v = .75$ and $u = .40$, and also with $v = .40$ and $u = .10$. In the former case, $v < p(E|H)$ and yet $p^*(H) = 65/109 > p(H)$. In the latter, it is even the case that $v < p_0(E|H)$, and yet $p^*(H) = 22/35 > p(H)$. In both cases the key inequality $\lambda_{p_1}(H, E) > \lambda_{p_0}(H, E)$ holds. Note 8 *supra* shows, correspondingly, that neither $v > p(E|H)$ nor $v > p_0(E|H)$ suffices to guarantee that $p^*(H) > p(H)$ for $p(E)$ sufficiently large.

As for the second possible objection, there is nothing incoherent about the fact that $p^*(E|H)$ differs from v. The judgment that H confers probability v on E is, as we have indicated, based only on theoretical considerations. It is not a final judgment about the conditional probability of E, given H, all things considered. The latter judgment is expressed by $p^*(E|H)$ and is based on both theory and observation. In the special case of implicational new explanation treated by Jeffrey, it follows from the fact that $p(H\overline{E}) = 0$ and from the fact that zeros are not raised by reparation (or, for that matter, by generalized reparation) that $p^*(E|H) = 1 = v$. But $p^*(E|H)$ can differ from v in the general case, where it is possible that $p(H\overline{E}) > 0$.

$E := \cup_{i\in J}E_i$. Certain observations have led to our assigning $p(E)$ a high value.

We subsequently discover, quite apart from the observations underlying p, theoretical grounds for believing that, for each i, the truth of H would confer probability v_i on E_i and, its falsity would confer probability u_i on E_i. Taken alone, these theoretical considerations would not have tended to alter whatever probability was ascribed to H prior to the assessment of p.

Proceeding as in §3.2, we resurrect an *ur*-distribution p_0, with p conceived as having come from p_0 by probability kinematics on $\{E_1, \ldots, E_n\}$. It is assumed that $p_0(HE_i) > 0$ and $p_0(\overline{H}E_i) > 0$ for every i. The aforementioned theoretical discovery would have led us to revise p_0 to the unique p_1 for which $p_1(E_i|H) = v_i$ and $p_1(E_i|\overline{H}) = u_i$ for all i, and $p_1(H) = p_0(H)$.

We now revise p to p^* in such a way that p^* bears the same relation to p as p_1 does to p_0, as dictated by the uniformity principle,

$$p^*(A) \propto p(A)p_1(A)/p_0(A),^{11} \tag{46}$$

for $A = HE_i, \overline{H}E_i$, where $i = 1, \ldots, n$. This determines p^* uniquely and this explanation-based revision of p raises the probability of H under conditions delineated in the following theorem.

Theorem 5. *Let J be a distinguished nonempty proper subset of $\{1, \ldots, n\}$, with $E := \cup_{i\in J}E_i$. If*

$$\min_{i\in J} \{\lambda_{p_1}(H,E_i)\} > \max_{i\in J} \{\lambda_{p_0}(H,E_i)\}, \tag{47}$$

then $p^*(H) > p(H)$ for sufficiently large values of $p(E)$.

Proof. It suffices to show that $p^*(H)/p^*(\overline{H}) > p(H)/p(\overline{H})$ for $p(E)$ sufficiently large. By the uniformity principle (46),

11. From (46) one gets the exact formula $p^*(A) = s^{-1}p(A)p_1(A)/p_0(A)$ for each $A \in \mathcal{A} := \{HE_1, \ldots, HE_n, \overline{H}E_1, \ldots, \overline{H}E_n\}$, where

$$s = \sum_{A\in\mathcal{A}} p(A)p_1(A)/p_0(A).$$

In a straightforward generalization of Theorem 2, one may prove that (46) is equivalent to either of the conditions

(i) $\beta_{p^*,p}(A_1:A_2) = \beta_{p_1,p_0}(A_1:A_2)$, for all $A_1, A_2 \in \mathcal{A}$, or

(ii) (a) $\beta_{p^*,p}(H:\overline{H}|E_i) = \beta_{p_1,p_0}(H:\overline{H}|E_i)$, for all $i = 1, \ldots, n$,

(b) $\beta_{p^*,p}(E_i:E_j|H) = \beta_{p_1,p_0}(E_i:E_j|H)$, for $i,j = 1, \ldots, n$, and

(c) $\beta_{p^*,p}(E_i:E_j|\overline{H}) = \beta_{p_1,p_0}(E_i:E_j|\overline{H})$, for $i,j = 1, \ldots, n$.

As in the case of Theorem 2, there is some redundance in (i) and in (ii).

$$p^*(H)/p^*(\overline{H}) = (t + \varepsilon)/(w + \delta), \tag{48}$$

where

$$t = \sum_{i\in J} p(HE_i)p_1(HE_i)/p_0(HE_i) = \sum_{i\in J} p(E_i)p_1(HE_i)/p_0(E_i), \tag{49}$$

$$\varepsilon = \sum_{i\notin J} p(HE_i)p_1(HE_i)/p_0(HE_i), \tag{50}$$

$$w = \sum_{i\in J} p(\overline{H}E_i)p_1(\overline{H}E_i)/p_0(\overline{H}E_i) = \sum_{i\in J} p(E_i)p_1(\overline{H}E_i)/p_0(E_i), \tag{51}$$

and

$$\delta = \sum_{i\notin J} p(\overline{H}E_i)p_1(\overline{H}E_i)/p_0(\overline{H}E_i). \tag{52}$$

Also,

$$p(H)/p(\overline{H}) = (t' + \varepsilon')/(w' + \delta'), \tag{53}$$

where

$$t' = \sum_{i\in J} p(E_i)p_0(H|E_i) = \sum_{i\in J} p(E_i)p_0(HE_i)/p_0(E_i), \tag{54}$$

$$\varepsilon' = \sum_{i\notin J} p(E_i)p_0(H|E_i), \tag{55}$$

$$w' = \sum_{i\in J} p(E_i)p_0(\overline{H}|E_i) = \sum_{i\in J} p(E_i)p_0(\overline{H}E_i)/p_0(E_i), \tag{56}$$

and

$$\delta' = \sum_{i\notin J} p(E_i)p_0(\overline{H}|E_i). \tag{57}$$

By (48) and (53), it suffices to show that

$$tw' - wt' > \delta t' + w\varepsilon' + \delta\varepsilon' - t\delta' - \varepsilon w' - \varepsilon\delta' \tag{58}$$

for $p(E)$ sufficiently large. Now δ, δ', ε, and ε' all converge to zero as $p(E) = \Sigma_{i\in J}p(E_i) \to 1$, and t, t', w, and w' are uniformly bounded above for all distributions p. Hence the right hand side of (58) converges to zero as $p(E) \to 1$. So to show that (58) obtains for $p(E)$ sufficiently large, it suffices to show the existence of a $\beta > 0$ such that $tw' - wt' \geq \beta$ for $p(E)$ sufficiently large.

Since $p_1(H) = p_0(H)$, condition (47) is equivalent to

$$p_1(HE_i)p_0(\overline{H}E_j) > p_1(\overline{H}E_i)p_0(HE_j) \text{ for all } i, j \in J. \tag{59}$$

Now by (49), (51), (54), and (56),

$$tw' - wt' = \sum_{i,j\in J} \frac{p(E_i)p(E_j)}{p_0(E_i)p_0(E_j)}[p_1(HE_i)p_0(\overline{H}E_j) - p_1(\overline{H}E_i)p_0(HE_j)]. \tag{60}$$

Let m be the minimum of the bracketed differences occurring in the above sum. By (59), $m > 0$. Then, for example, whenever $p(E) \geq 1/2$, it follows from (60) that $tw' - wt' \geq \beta = m/4$. □

Theorem 5 reduces to Theorem 3 when $n = 2$ and $|J| = 1$. Using essentially the same proof as that of Theorem 4, one may show that p^* comes from p_1 by probability kinematics on $\{E_1, \ldots, E_n\}$, with

$$\beta_{p^*,p_1}(E_i:E_j) = \beta_{p,p_0}(E_i:E_j) \tag{61}$$

for all $i, j \in \{1, \ldots, n\}$. So the explanation-based revision p^* of p may be derived from the commutativity principle as well as from the uniformity principle.

REFERENCES

Garber, D. (1983), Old evidence and logical omniscience", in J. Earman (ed.), *Testing Scientific Theories.* Minnesota Studies in the Philosophy of Science X. Minneapolis: University of Minnesota Press, pp. 99–131.

Glymour, C. (1980), *Theory and Evidence.* Princeton: Princeton University Press.

Jeffrey, R. (1983), *The Logic of Decision,* 2nd edition. Chicago: University of Chicago Press.

———. (1988), "Conditioning, kinematics, and exchangeability", in B. Skyrms and W. Harper (eds.), *Causation, Chance, and Credence,* vol. 1. Boston: Kluwer Academic Publishers, pp. 221–255.

———. (1991), "Postscript 1991: New explanation revisited", in Jeffrey 1992, 103–107.

———. (1992), *Probability and the Art of Judgement.* Cambridge: Cambridge University Press.

———. (1995), "Probability reparation: the problem of new explanation", *Philosophical Studies* 77:97–102.

Weinberg, S. (1992), *Dreams of a Final Theory.* New York: Pantheon Press.

Zynda, L. (1995), "Old evidence and new theories," *Philosophical Studies* 77:67–96.