ACTA ARITHMETICA
XVII (1971)

Interpolation series for continuous functions on π-adic completions of $\mathrm{GF}(q, x)$*

by

CARL G. WAGNER (Knoxville, Tenn.)

1. Introduction. In 1944 Dieudonné [7] proved an analogue of the Weierstrass Approximation Theorem for continuous functions of a p-adic variable. In 1958 Mahler [8] sharpened this result by exhibiting a series expansion for continuous functions defined on the p-adic integers. He showed that every such function f is the uniform limit of an interpolation series

$$f(t) = \sum_{n=0}^{\infty} A_n \binom{t}{n} \tag{1.1}$$

where the coefficients A_n are uniquely determined by

$$A_n = \Delta^n f(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k). \tag{1.2}$$

In the present paper we choose an irreducible element π from the polynomial ring $\mathrm{GF}[q, x]$ over the finite field $\mathrm{GF}(q)$ and use it to equip the function field $\mathrm{GF}(q, x)$ with a π-adic absolute value. We denote by F_π the completion of $\mathrm{GF}(q, x)$ for this absolute value and by I_π the valuation ring of F_π. The aforementioned theorem of Dieudonné may easily be seen to generalize to the case of a locally compact non-archimedean field. Hence, every continuous function $f\colon K \to F_\pi$, where K is a compact subset of F_π, is the uniform limit of some sequence of polynomials over F_π. Our aim in this paper is to prove some Mahler type theorems for such functions.

We mention that Amice [1] has already constructed a certain type of series approximation for continuous functions defined on locally compact non-archimedean fields. In the process, Amice characterized those sequences ("suites très bien réparties") in the domain of a continuous function with respect to which a Newton type interpolation procedure will yield

* This research was supported in part by the National Science Foundation, under Research Grant GP — 7855.

a uniformly convergent series approximation for that function. In particular, the nonnegative rational integers, ordered in the usual way, constitute such a sequence in the p-adic integers, and so Mahler's result appears as a special case of Amice's Interpolation Theorem [1].

In what follows, we exhibit a "suite très bien répartie" in I_π, denoted $\{m_i\}$, consisting of a special sequential ordering of $GF[q, x]$. Specializing Amice, we prove (Theorem 4.4) that for every continuous function $f\colon I_\pi \to F_\pi$ there exists a unique sequence $\{A_i\}$ in F_π such that

$$f(t) = \sum_{i=0}^{\infty} A_i Q_i(t), \tag{1.3}$$

where $Q_i(t)$ is the ith Newton interpolation polynomial for the interpolation sequence $\{m_i\}$, and (1.3) converges uniformly on I_π. We add that $\{A_i\}$ is always a null sequence, i.e., $\lim_{i\to\infty} A_i = 0$. Moreover, the above result may be extended to continuous functions $f\colon K \to F_\pi$, where K is any compact subset of F_π, by employing a Urysohn type theorem for totally disconnected spaces due to Dieudonné [7].

We may regard the foregoing approach to constructing function field analogues of Mahler's result as deriving from the observation that the polynomials $\binom{t}{n}$ are the Newton interpolation polynomials for the nonnegative rational integers. From this standpoint, the crucial problem, completely solved by Amice, is that of identifying those sequences in I_π for which the associated Newton polynomials yield interpolation series for continuous functions.

If, instead, one regards the sequence $\left\{\binom{t}{n}\right\}$ merely as an ordered basis of the Q_p-vector space $Q_p[t]$, then one is led to ask which ordered bases of the F_π-vector space $F_\pi[t]$ yield interpolation series for continuous functions on I_π. In this connection, it is of interest to recall that the sequence $\left\{\binom{t}{n}\right\}$ has the further property of being an ordered basis of the Z-module of polynomials over Q which map Z into Z (and also of the Z_p-module of polynomials over Q_p that map Z_p into Z_p, where Z_p is the valuation ring of Q_p).

The function field analogue of the latter property is that of being an ordered basis of the I_π-module of polynomials over F_π that map I_π into I_π. Let $\{H_i(t)\}$ be such a basis. We prove (Theorem 4.5) that for every continuous function $f\colon I_\pi \to I_\pi$ there exists a unique null sequence $\{B_i\}$ in I_π such that

$$f(t) = \sum_{i=0}^{\infty} B_i H_i(t), \tag{1.4}$$

where (1.4) converges uniformly on I_π.

The above theorem may be applied to a sequence of polynomials $\{G_i(t)/g_i\}$ introduced in 1948 by Carlitz [4]. This leads to the following characterization (Theorem 5.1) of continuous linear operators on the $\mathrm{GF}(q)$-vector space I_π: Let $f\colon I_\pi \to I_\pi$ be continuous. If the (unique) interpolation series for f constructed from the Carlitz polynomials is given by

$$f(t) = \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i}, \tag{1.5}$$

then f is a linear operator on the $\mathrm{GF}(q)$-vector space I_π if and only if $A_i = 0$ for $i \neq q^k$, where $k \geqslant 0$.

The author wishes to thank Professor Carlitz for his guidance and encouragement during the preparation of this paper, and of the doctoral dissertation on which it is based.

2. Preliminaries. Let $\mathrm{GF}(q)$ be a finite field of cardinality q. Denote by $\mathrm{GF}[q, x]$ the ring of polynomials in an indeterminate x over $\mathrm{GF}(q)$, and by $\mathrm{GF}(q, x)$ the quotient field of $\mathrm{GF}[q, x]$. Let $\pi \in \mathrm{GF}[q, x]$ be an irreducible polynomial of degree d. Then every nonzero $\alpha \in \mathrm{GF}(q, x)$ may be written, in essentially unique fashion,

$$\alpha = \pi^n \frac{m_1}{m_2}, \tag{2.1}$$

where n is integral, and m_1 and m_2 are polynomials prime to each other and to π.

Define a function $v_\pi\colon \mathrm{GF}(q, x) - \{0\} \to Z$ by

$$v_\pi(\alpha) = n, \tag{2.2}$$

where α is written as in (2.1). It follows that

$$v_\pi(\alpha\beta) = v_\pi(\alpha) + v_\pi(\beta) \qquad (\alpha\beta \neq 0) \tag{2.3}$$

and

$$v_\pi(\alpha+\beta) \geqslant \min\{v_\pi(\alpha), v_\pi(\beta)\} \qquad (\alpha, \beta, \alpha+\beta \neq 0). \tag{2.4}$$

Fixing a real number b such that $0 < b < 1$, define the π-adic absolute value $|\ |_\pi$ on $\mathrm{GF}(q, x)$ as follows:

$$|0|_\pi = 0, \tag{2.5}$$

$$|\alpha|_\pi = b^{v_\pi(\alpha)} \qquad (\alpha \neq 0). \tag{2.6}$$

By familiar methods $\mathrm{GF}(q, x)$ may be embedded as a dense subfield in an essentially unique complete field, denoted F_π. With respect to the extended absolute value, F_π is a discrete non-archimedean field. Equipped with the metric d_π, defined by

(2.7) $$d_\pi(\alpha, \beta) = |\alpha - \beta|_\pi,$$

F_π is a metric field. In particular, polynomial functions over F_π are continuous.

Denote by I_π the valuation ring of F_π, i.e.,

$$I_\pi = \{\alpha \in F_\pi \colon |\alpha|_\pi \leqslant 1\}.$$

Then the valuation ideal

$$(\pi) = \{\alpha \in I_\pi \colon |\alpha|_\pi < 1\}$$

is maximal and the residue class field $I_\pi/(\pi)$ is isomorphic to $\mathrm{GF}(q^d)$, where $d = \deg \pi$.

Let Γ be a complete set of representatives of $I_\pi/(\pi)$ in I_π. Then every nonzero $\alpha \in F_\pi$ may be uniquely represented as a π-series,

(2.8) $$\alpha = \pi^n \sum_{i=0}^{\infty} a_i \pi^i,$$

where $a_i \in \Gamma$, $\pi \nmid a_0$ in I_π, and $|\alpha| = b^n$ [6]. In particular, Γ may be taken to be the set of polynomials in $\mathrm{GF}[q, x]$ having degree less than d.

For $\alpha \in F_\pi$ and k any integer, let

(2.9) $$B_k(\alpha) = \{\beta \in F_\pi \colon |\beta - \alpha|_\pi \leqslant b^k\} = \{\beta \in F_\pi \colon |\beta - \alpha|_\pi < b^{k-1}\}.$$

Then the collection $\{B_k(\alpha) \colon k \geqslant 0\}$ is a fundamental system of open-closed neighborhoods of α; hence F_π is totally disconnected.

Again, let Γ be a complete set of representatives of $I_\pi/(\pi)$ in I_π. Given $\varepsilon > 0$, let k be a positive integer such that $b^k < \varepsilon$. Let

(2.10) $$\Delta = \{\alpha \in I_\pi \colon \alpha = a_0 + a_1\pi + \ldots + a_{k-1}\pi^{k-1}\}$$

where $a_i \in \Gamma$. Then Δ has q^{kd} elements and the collection

(2.11) $$\{B_k(\alpha) \colon \alpha \in \Delta\}$$

is a pairwise disjoint open cover of I_π, all of the members of which have radius less than ε. It follows that I_π (and, therefore, every closed and bounded subset of F_π) is compact. (In fact, the Heine–Borel Theorem holds in all locally compact non-archimedean fields, a result due to Schöbe [9].)

In the special case $\pi = x$, the complete field F_x may be identified with the field of formal power series over $\mathrm{GF}(q)$, for by (2.8) every nonzero $\alpha \in F_x$ may be written

(2.12) $$\alpha = \sum_{i=-\infty}^{\infty} a_i x^i,$$

where $a_i \in \mathrm{GF}(q)$, all but a finite number of the a_i vanish for $i < 0$, and $|a|_x = b^n$, for n the smallest integer such that $a_n \neq 0$.

There would, in fact, be no loss of generality in restricting the investigation we have in mind to the case of x-adic absolute values; for it is known that every locally compact Hausdorff field having nonzero characteristic is topologically isomorphic to a field of formal power series in one indeterminate over some finite field ([10], pp. 12–22). In the case of the fields F_π we may specialize this result as follows.

THEOREM 2.1. *Let F_π be the completion of* $\mathrm{GF}(q, x)$ *for the absolute value* $|\ |_\pi$, *where π is an irreducible polynomial of degree d. Then F_π is topologically isomorphic to a field of formal power series in one indeterminate over the finite field* $\mathrm{GF}(q^d)$.

Proof. In view of representations (2.8) and (2.12), it suffices to show that Γ, a complete set of representatives of $I_\pi/(\pi)$ in I_π, may be chosen in such a way that Γ is a *subfield* of I_π.

Let $a \in I_\pi$. Since $I_\pi/(\pi)$ is isomorphic to $\mathrm{GF}(q^d)$, it follows that $\pi \mid a^{q^d} - a$, and hence that

$$\pi^{q^{(n-1)d}} \mid a^{q^{nd}} - a^{q^{(n-1)d}},$$

for all natural numbers n. Therefore, the series

$$a + (a^{q^d} - a) + (a^{q^{2d}} - a^{q^d}) + \ldots \tag{2.13}$$

converges, i.e., $\lim\limits_{n\to\infty} a^{q^{nd}}$ exists for all $a \in I_\pi$.

Define a function $w\colon I_\pi \to I_\pi$ by

$$w(a) = \lim_{n\to\infty} a^{q^{nd}}. \tag{2.14}$$

Then w is an endomorphism of the ring I_π with kernel (π), and so $w(I_\pi)$ is a subfield of I_π isomorphic to $\mathrm{GF}(q^d)$. By (2.13) and (2.14), it follows that $w(a) \equiv a (\operatorname{mod} \pi)$; hence we may take $\Gamma = w(I_\pi)$, as desired.

To conclude this section, we recall that, in addition to the π-adic absolute values, $\mathrm{GF}(q, x)$ admits only one other non-trivial absolute value, $|\ |_\infty$, defined by

$$\left|\frac{m_1}{m_2}\right|_\infty = b^{\deg m_2 - \deg m_1}, \tag{2.15}$$

for m_1, m_2 nonzero elements of $\mathrm{GF}[q, x]$ and $0 < b < 1$ ([6], pp. 45–47). The completion of $\mathrm{GF}(q, x)$ for $|\ |_\infty$, denoted by F_∞, may be seen to consists of the set of all descending formal power series over $\mathrm{GF}(q)$,

$$a = \sum_{i=-\infty}^{\infty} a_i x^{-i}, \tag{2.16}$$

where $a_i \in \mathrm{GF}(q)$, all but a finite number of these coefficients vanish for $i < 0$, and $|a|_\infty = b^n$, n the smallest integer such that $a_n \neq 0$.

In what follows, we shall appeal to the obvious topological isomorphism between F_x and F_∞ to omit an explicit treatment of the problem of approximating continuous functions in F_∞. There appears, however, to be no particular advantage in a similar appeal to Theorem 2.1, and so we shall state our results for the fields F_π.

3. A special ordering of $\mathrm{GF}[q, x]$. Let $\pi \in \mathrm{GF}[q, x]$ be an irreducible polynomial of degree d. We define a sequential ordering of $\mathrm{GF}[q, x]$ which has the property of being, in the terminology of Amice [1], "très bien répartie" in I_π. Let $(a_0, a_1, \ldots, a_{q^d-1})$ be a fixed ordering of the polynomials in $\mathrm{GF}[q, x]$ of degree $< d$ such that $a_0 = 0$, $a_1 = 1$, and $\deg a_i \leqslant \deg a_j$ for $1 \leqslant i \leqslant j$. The special sequence $\{m_n\}$, running through $\mathrm{GF}[q, x]$, is defined as follows. If

$$n = k_0 + k_1 q^d + \ldots + k_s q^{sd} \qquad (0 \leqslant k_i < q^d), \tag{3.1}$$

set

$$m_n = a_{k_0} + a_{k_1}\pi + \ldots + a_{k_s}\pi^s. \tag{3.2}$$

THEOREM 3.1. *For any integers $s \geqslant 0$ and $k \geqslant 1$, the set*

$$\{m_{i+sq^{kd}}:\ 0 \leqslant i < q^{kd}\} \tag{3.3}$$

is a complete residue system $(\mathrm{mod}\,\pi^k)$.

Proof. As there is no "overlap" in the q^d-adic expansions (3.1) of i and sq^{kd}, it follows that

$$m_{i+sq^{kd}} = m_i + m_{sq^{kd}}. \tag{3.4}$$

The set $\{m_i:\ 0 \leqslant i < q^{kd}\}$ is a complete residue system $(\mathrm{mod}\,\pi^k)$, and this property is preserved under shifting by the additive constant $m_{sq^{kd}}$.

Let

$$S_n = \{m_0, m_1, \ldots, m_{n-1}\} \qquad (n \geqslant 1), \tag{3.5}$$

and let

$$\varrho(a; k, n) = \mathrm{card}\big(B_k(a) \cap S_n\big), \tag{3.6}$$

with $a \in I_\pi$, $n, k \geqslant 1$, and $B_k(a)$ as in (2.9). Then the following theorem is a straightforward consequence of Theorem 3.1.

THEOREM 3.2. *For every $a \in I_\pi$, and for all positive integers n and k,*

$$\left[\frac{n}{q^{kd}}\right] \leqslant \varrho(a; k, n) \leqslant \left[\frac{n-1}{q^{kd}}\right] + 1. \tag{3.7}$$

Furthermore,

$$\varrho(m_n; k, n) = \left[\frac{n}{q^{kd}}\right]. \tag{3.8}$$

We now introduce an ordered basis of the F_π-vector space $F_\pi[t]$, consisting of the Newton interpolation polynomials for the interpolation sequence $\{m_n\}$, defined by (3.2). Set

$$P_0(t) = 1, \quad P_n(t) = (t-m_0)(t-m_1)\ldots(t-m_{n-1}) \quad (n \geqslant 1), \tag{3.9}$$

and

$$Q_0(t) = 1, \quad Q_n(t) = P_n(t)/P_n(m_n) \quad (n \geqslant 1). \tag{3.10}$$

Since $\deg Q_n(t) = n$, $\{Q_n(t)\}$ is an ordered basis of the F_π-vector space $F_\pi[t]$. Hence, every polynomial $g(t) \in F_\pi[t]$ of degree $\leqslant n$ may be written uniquely as

$$g(t) = \sum_{i=0}^{n} A_i Q_i(t). \tag{3.11}$$

To derive a formula for the coefficients A_i, let $g_r(t)$ be the unique polynomial of degree $\leqslant r$ for which $g_r(m_j) = g(m_j)$ for $0 \leqslant j \leqslant r$. Then

$$g_r(t) = \sum_{j=0}^{r} A_j Q_j(t) = \sum_{j=0}^{r} \frac{P_{r+1}(t)g(m_j)}{(t-m_j)P'_{r+1}(m_j)}, \tag{3.12}$$

where the second equality above is the result of Lagrange interpolation. It follows from (3.12) that

$$g_i(t) - g_{i-1}(t) = A_i Q_i(t) = \left(P_i(m_i)\sum_{j=0}^{i}\frac{g(m_j)}{P'_{i+1}(m_j)}\right)Q_i(t). \tag{3.13}$$

Hence

$$A_i = P_i(m_i)\sum_{j=0}^{i}\frac{g(m_j)}{P'_{i+1}(m_j)}. \tag{3.14}$$

The following two theorems imply that the sequence $\{Q_n(t)\}$ is, in fact, an ordered basis of the I_π-module of polynomials over F_π that map I_π into itself. In the remainder of the paper the subscript π will be omitted from the symbols v_π and $|\ |_\pi$.

THEOREM 3.3. *For all* $t \in I_\pi$, $|Q_n(t)| \leqslant 1$.

Proof (Amice [1]). In virtue of (2.6) it suffices to show that

$$v(P_n(t)) \geqslant v(P_n(m_n)). \tag{3.15}$$

By (3.7),

$$(3.16)\qquad v(P_n(t)) = \sum_{i=0}^{n-1} v(t-m_i) = \sum_{k=1}^{\infty} k\big(\varrho(t;k,n)-\varrho(t;k+1,n)\big)$$
$$= \sum_{k=1}^{\infty} \varrho(t;k,n) \geqslant \sum_{k=1}^{\infty} \left[\frac{n}{q^{kd}}\right].$$

But, by (3.8),

$$(3.17)\qquad v(P_n(m_n)) = \sum_{k=1}^{\infty} \varrho(m_n;k,n) = \sum_{k=1}^{\infty} \left[\frac{n}{q^{kd}}\right],$$

from which the desired result follows.

THEOREM 3.4. *Let* $g(t)\in F_\pi[t]$, *and write*

$$(3.18)\qquad g(t) = \sum_{i=0}^{n} A_i Q_i(t).$$

Then g *maps* I_π *into itself if and only if* $A_i \in I_\pi$, *for* $0\leqslant i\leqslant n$.

Proof. Sufficiency. By Theorem 3.3, $|Q_i(t)|\leqslant 1$ if $|t|\leqslant 1$, so if $|A_i|\leqslant 1$, $|g(t)|\leqslant 1$, since $|\ |$ is non-archimedean.

Necessity. By (3.14), it suffices to show that, for all $j\leqslant i$,

$$(3.19)\qquad v(P_i(m_i)) \geqslant v(P'_{i+1}(m_j)).$$

By (3.17),

$$(3.20)\qquad v(P_i(m_i)) = \sum_{k=1}^{\infty}\left[\frac{i}{q^{kd}}\right].$$

We show that

$$(3.21)\qquad v(P'_{i+1}(m_i)) \leqslant \sum_{k=1}^{\infty}\left[\frac{i}{q^{kd}}\right].$$

Since

$$(3.22)\qquad P'_{i+1}(m_j) = (m_j-m_0)\ldots(m_j-m_{j-1})(m_j-m_{j+1})\ldots(m_j-m_i),$$

inequality (3.21) is obvious for $j=i$, so assume that $j<i$. Denote by $S(i,j)$ the set $S_{i+1}-\{m_j\}$. Then

$$(3.23)\qquad v(P'_{i+1}(m_j)) = \sum_{\substack{r=0\\ r\neq j}}^{i} v(m_j-m_r)$$
$$= \sum_{k=1}^{\infty} k\big(\operatorname{card}(B_k(m_j)\cap S(i,j)) - \operatorname{card}(B_{k+1}(m_j)\cap S(i,j))\big)$$
$$= \sum_{k=1}^{\infty} \operatorname{card}(B_k(m_j)\cap S(i,j)) \leqslant \sum_{k=1}^{\infty}\left[\frac{i}{q^{kd}}\right],$$

as desired.

4. Interpolation theorems. We require a preliminary theorem, due to Amice [1], which specifies conditions under which certain finite subsets of $\{Q_i(t)\}$ are locally constant $(\mathrm{mod}\,\pi)$. As in the case of a previous theorem, we include, for completeness, a specialized version of the proof given by Amice.

THEOREM 4.1. *Let $\pi \in \mathrm{GF}[q, x]$ be an irreducible polynomial of degree d, and let $|\pi| = b$. Then, for all $k \geqslant 1$ and for all i such that $0 \leqslant i \leqslant q^{kd}-1$, if $t_1, t_2 \in I_\pi$ and $|t_1 - t_2| \leqslant b^k$, then*

$$|Q_i(t_1) - Q_i(t_2)| \leqslant b.$$

Proof. It suffices to show that for all i, j with $0 \leqslant i, j \leqslant q^{kd}-1$, if $t \in B_k(m_j)$, then $|Q_i(t) - Q_i(m_j)| \leqslant b$. The cases (1) $j < i$ and (2) $j \geqslant i$ are treated separately.

(1) If $j < i$, then $|Q_i(t) - Q_i(m_j)| = |Q_i(t)|$, and so it suffices to show that, for $t \in B_k(m_j)$,

$$v(t-m_0)+\ldots+v(t-m_{i-1}) > v(m_i-m_0)+\ldots+v(m_i-m_{i-1}), \tag{4.1}$$

or, as in (3.16), that

$$\sum_{r=1}^{\infty} \varrho(t; r, i) > \sum_{r=1}^{\infty} \varrho(m_i; r, i). \tag{4.2}$$

By Theorem 3.2,

$$\varrho(t; r, i) \geqslant \varrho(m_i; r, i) = \left[\frac{i}{q^{rd}}\right]. \tag{4.3}$$

When $r = k$, however, inequality (4.3) is strict, since

$$\varrho(t; k, i) = 1 \quad \text{and} \quad \varrho(m_i; k, i) = 0.$$

(2) Let $i \leqslant j$. By hypothesis, $|t - m_j| \leqslant b^k$. For all r with $0 \leqslant r \leqslant i-1 < j$, $m_r \not\equiv m_j (\mathrm{mod}\,\pi^k)$, and so

$$|t-m_j| \leqslant b\,|m_j - m_r|, \tag{4.4}$$

or

$$|(t-m_r)-(m_j-m_r)| \leqslant b\,|m_j-m_r|, \tag{4.5}$$

or

$$\left|\frac{t-m_r}{m_j-m_r} - 1\right| \leqslant b. \tag{4.6}$$

Hence, for each r, there is an $a_r \in I_\pi$ such that

$$\frac{t-m_r}{m_j-m_r} = 1 + \pi a_r, \tag{4.7}$$

and so, there is a $\beta \in I_\pi$ such that

$$\prod_{r=0}^{i-1} \frac{t-m_r}{m_j-m_r} = \frac{Q_i(t)}{Q_i(m_j)} = 1+\pi\beta. \tag{4.8}$$

Therefore,

$$\left|\frac{Q_i(t)}{Q_i(m_j)} - 1\right| \leqslant b, \tag{4.9}$$

and, by Theorem 3.3,

$$|Q_i(t)-Q_i(m_j)| \leqslant b\,|Q_i(m_j)| \leqslant b. \tag{4.10}$$

The interpolation theorems announced in the Introduction are included in the following sequence of theorems.

THEOREM 4.2. *Let π, b, and d be as in Theorem 4.1. Let f: $I_\pi \to I_\pi$ be continuous. Then there is an integer $k \geqslant 1$ and a continuous function h: $I_\pi \to I_\pi$ such that*

$$f(t) = \sum_{i=0}^{q^{kd}-1} f(m_i)\chi_i(t) + \pi h(t), \tag{4.11}$$

where χ_i is the characteristic function of the set $B_k(m_i)$.

Proof. Since I_π is compact, f is uniformly continuous. Hence, there is an integer $k \geqslant 1$ such that, for all i, $0 \leqslant i \leqslant q^{kd}-1$, if $t \in B_k(m_i)$, then $|f(t)-f(m_i)| \leqslant b$. Thus, there is a continuous function h^i: $B_k(m_i) \to I_\pi$ such that, for $t \in B_k(m_i)$,

$$f(t) = f(m_i) + \pi h^i(t). \tag{4.12}$$

Since the sets $B_k(m_i)$ are a pairwise disjoint open-closed cover of I_π, (4.11) may be gotten by setting $h(t) = h^i(t)$ for $t \in B_k(m_i)$.

THEOREM 4.3. *Let f: $I_\pi \to I_\pi$ be continuous. Then there is an integer $k \geqslant 1$, a continuous function f_1: $I_\pi \to I_\pi$, and a sequence $\{\alpha_i\colon 0 \leqslant i \leqslant q^{kd}-1\}$ in I_π such that*

$$f(t) = \sum_{i=0}^{q^{kd}-1} \alpha_i Q_i(t) + \pi f_1(t). \tag{4.13}$$

Proof. Using the uniform continuity of f, determine k as in Theorem 4.2. By Theorem 4.1, this k is also associated with the uniform continuity of the functions $Q_i(t)$, $0 \leqslant i \leqslant q^{kd}-1$. Applying Theorem 4.2 to these functions, we get

$$Q_i(t) = \sum_{j=0}^{q^{kd}-1} Q_i(m_j)\chi_j(t) + \pi h_i(t). \tag{4.14}$$

Since $Q_i(m_j) = 0$ when $j < i$, system (4.14) is triangular. Solving for the functions $\chi_i(t)$ in terms of the $Q_i(t)$ and the error functions $h_i(t)$, and substituting in (4.11), we get (4.13), where $f_1(t)$ is expressed in terms of the error functions $h_i(t)$.

THEOREM 4.4. *Let $f\colon I_\pi \to I_\pi$ be continuous. Then there is a unique sequence $\{A_i\}$ in F_π such that*

$$f(t) = \sum_{i=0}^{\infty} A_i Q_i(t), \tag{4.15}$$

where (4.15) *converges uniformly on I_π. Moreover, for all i, $|A_i| \leqslant 1$ and $\lim\limits_{i\to\infty} A_i = 0$.*

Proof. By Theorem 4.3 there is an integer $k_0 \geqslant 1$, a sequence $\{a_i^0\colon 0 \leqslant i \leqslant q^{k_0 d} - 1\}$, and a continuous function $f_1\colon I_\pi \to I_\pi$ such that

$$f(t) = \sum_{i=0}^{q^{k_0 d}-1} a_i^0 Q_i(t) + \pi f_1(t). \tag{4.16}$$

Similarly, we may write

$$f_1(t) = \sum_{i=0}^{q^{k_1 d}-1} a_i^1 Q_i(t) + \pi f_2(t). \tag{4.17}$$

Iterating and substituting in (4.16) at each stage, we get

$$f(t) = \sum_{i=0}^{M_{n-1}-1} (a_i^0 + \pi a_i^1 + \ldots + \pi^{n-1} a_i^{n-1}) Q_i(t) + \pi f_n(t), \tag{4.18}$$

where

$$M_{n-1} = \max\{q^{k_0 d}, q^{k_1 d}, \ldots, q^{k_{n-1} d}\}. \tag{4.19}$$

Define the sequence $\{A_i\}$ by

$$A_i = \sum_{j=0}^{\infty} \pi^j a_i^j. \tag{4.20}$$

The series (4.20) converges to an element of I_π, for $|a_i^j| \leqslant 1$. Also

$$\lim_{i\to\infty} A_i = 0, \tag{4.21}$$

for if $i \geqslant M_{n-1}$, then $a_i^0 = a_i^1 = \ldots = a_i^{n-1} = 0$, and so $|A_i| \leqslant b^n$.

Let $k \geqslant M_{n-1} - 1$. Then

$$\Big|\sum_{i=0}^{k} A_i Q_i(t) - \sum_{i=0}^{M_{n-1}-1} (a_i^0 + \pi a_i^1 + \ldots + \pi^{n-1} a_i^{n-1}) Q_i(t)\Big| \tag{4.22}$$

$$\leqslant \max\Big\{\Big|\sum_{i=M_{n-1}}^{k} A_i Q_i(t)\Big|, \Big|\sum_{i=0}^{M_{n-1}-1} \big(A_i - (a_i^0 + \ldots + \pi^{n-1} a_i^{n-1})\big) Q_i(t)\Big|\Big\} \leqslant b^n,$$

and by (4.18)

$$\left| f(t) - \sum_{i=0}^{k} A_i Q_i(t) \right| \leqslant b^n. \tag{4.23}$$

Hence (4.15) converges uniformly to f on I_π. The coefficients A_i are uniquely determined by f, since for each $n \geqslant 0$, the finite sum

$$\sum_{i=0}^{n} A_i Q_i(t) \tag{4.24}$$

is the unique polynomial of degree $\leqslant n$ which takes the same values as f on the set $\{m_0, \ldots, m_n\}$. Hence, by (3.14),

$$A_i = P_i(m_i) \sum_{j=0}^{i} \frac{f(m_j)}{P'_{i+1}(m_j)}. \tag{4.25}$$

In the slightly more general case of a continuous function $f\colon I_\pi \to F_\pi$, the boundedness of f implies the existence of an integer $k \geqslant 0$ such that $\pi^k f\colon I_\pi \to I_\pi$. Hence

$$\pi^k f(t) = \sum_{i=0}^{\infty} \left(P_i(m_i) \sum_{j=0}^{i} \frac{\pi^k f(m_j)}{P'_{i+1}(m_j)} \right) Q_i(t), \tag{4.26}$$

and so

$$f(t) = \sum_{i=0}^{\infty} A_i Q_i(t), \tag{4.27}$$

where A_i is defined by (4.25).

In the case of a continuous function $f\colon B_k(0) \to F_\pi$, where $k < 0$, define $g\colon I_\pi \to F_\pi$ by $g(t) = f(\pi^k t)$. Then by (4.21) and (4.27), we have, for all $t \in I_\pi$,

$$f(\pi^k t) = g(t) = \sum_{i=0}^{\infty} \left(P_i(m_i) \sum_{j=0}^{i} \frac{f(\pi^k m_j)}{P'_{i+1}(m_j)} \right) Q_i(t). \tag{4.28}$$

Hence, for all $t \in B_k(0)$,

$$f(t) = f\big(\pi^k(\pi^{-k} t)\big) = \sum_{i=0}^{\infty} \left(P_i(m_i) \sum_{j=0}^{i} \frac{f(\pi^k m_j)}{P'_{i+1}(m_j)} \right) Q_i(\pi^{-k} t). \tag{4.29}$$

It follows that every continuous function $f\colon K \to F_\pi$, where K is a compact subset of F_π, has a series expansion of the form (4.29), for $K \subseteq B_k(0)$ for some $k \leqslant 0$ and, by a theorem of Dieudonné ([7], p. 82), any such f has a continuous extension to $B_k(0)$.

THEOREM 4.5. *Let $\{H_i(t)\}$ be an ordered basis of the I_π-module of polynomials over F_π that map I_π into itself. Let $f\colon I_\pi \to I_\pi$ be continuous. Then there exists a unique null sequence $\{B_i\}$ in I_π such that*

$$f(t) = \sum_{i=0}^{\infty} B_i H_i(t), \tag{4.30}$$

where (4.30) *converges uniformly on I_π.*

Proof. By Theorem 4.4,

$$f(t) = \sum_{j=0}^{\infty} A_j Q_j(t), \tag{4.31}$$

where $A_j \in I_\pi$ and $\lim_{j\to\infty} A_j = 0$. By Theorem 3.3, for all $j \geqslant 0$, $Q_j(t)$ may be written uniquely as

$$Q_j(t) = \sum_{i=0}^{n_j} D_i^j H_i(t), \tag{4.32}$$

where $D_i^j \in I_\pi$. Set

$$B_i = \sum_{j=0}^{\infty} A_j D_i^j. \tag{4.33}$$

Since $\lim_{j\to\infty} A_j = 0$ and $|D_i^j| \leqslant 1$, (4.33) converges to an element of I_π. Moreover, $\lim_{i\to\infty} B_i = 0$, for, given any integer $k \geqslant 0$, let r be such that $|A_j| \leqslant b^k$ if $j \geqslant r$. Let $i > \max\{n_0, \ldots, n_{r-1}\}$. Then $D_i^j = 0$ if $j < r$, and so $|B_i| \leqslant b^k$.

If $k \geqslant 0$, let r be such that $|A_j| \leqslant b^k$ for $j \geqslant r$ and

$$\Big|\sum_{j=0}^{s} A_j Q_j(t) - f(t)\Big| \leqslant b^k \tag{4.34}$$

for $s \geqslant r$. If $n \geqslant \max\{n_0, \ldots, n_r\}$, then

$$\Big|\sum_{i=0}^{n} B_i H_i(t) - \sum_{j=0}^{r} A_j Q_j(t)\Big| = \Big|\sum_{u=r+1}^{\infty} A_u \sum_{v=0}^{n} D_v^u H_v(t)\Big| \leqslant b^k. \tag{4.35}$$

Then (4.34) and (4.35) yield (4.30).

Moreover, $\{B_i\}$, as defined in (4.33), is the only null sequence in I_π for which (4.30) holds. For suppose that

$$f(t) = \sum_{i=0}^{\infty} C_i H_i(t), \tag{4.36}$$

where $\lim_{i\to\infty} C_i = 0$. For all $i \geqslant 0$, write

$$H_i(t) = \sum_{j=0}^{n_i} E_j^i Q_j(t), \tag{4.37}$$

where $E_j^i \in I_\pi$. A repetition of the preceding argument yields

$$f(t) = \sum_{j=0}^{\infty} Q_j(t) \sum_{i=0}^{\infty} C_i E_j^i. \tag{4.38}$$

By Theorem 4.4, however,

$$\sum_{i=0}^{\infty} C_i E_j^i = A_j \quad (j \geqslant 0), \tag{4.39}$$

where A_j is defined by (4.25). Since $\{C_i\}$ and $\{A_j\}$ are null sequences, the equations (4.39) may be written matrically,

$$MC = A, \tag{4.40}$$

where C and A are the infinite column vectors $[C_0, C_1, \ldots]^T$ and $[A_0, A_1, \ldots]^T$ and M is the column-finite matrix $[m_{rs}]$, where

$$m_{rs} = E_r^s \quad (r, s \geqslant 0), \tag{4.41}$$

and E_r^s is defined by (4.39).

Using (4.32) and (4.37) the matrix M may be seen to possess the two-sided inverse $Q = [q_{rs}]$, where

$$q_{rs} = D_r^s \quad (r, s \geqslant 0), \tag{4.42}$$

and D_r^s is defined by (4.32). Hence the relation (4.40) determines C uniquely, and

$$C_i = B_i = \sum_{j=0}^{\infty} A_j D_i^j. \tag{4.43}$$

We stress that Theorem 4.5 asserts the uniqueness of the coefficients B_i on the assumption that $\{B_i\}$ is null. The unqualified uniqueness of these coefficients (which we have been able to prove only in special cases) is equivalent to the assertion that a series

$$\sum_{i=0}^{\infty} C_i H_i(t) \tag{4.44}$$

converges uniformly on I_π only if $\{C_i\}$ is null.

5. Applications. Define the sequence of polynomials $\psi_r(t)$ over $GF[q, x]$ by

$$\psi_r(t) = \prod_{\deg m < r} (t - m), \quad \psi_0(t) = t, \tag{5.1}$$

where the product in (5.1) extends over all polynomials $m \in GF[q, x]$ (including 0) having degree $< r$. It follows [3] that

$$\psi_r(t) = \sum_{i=0}^{r} (-1)^{r-i} \begin{bmatrix} r \\ i \end{bmatrix} t^{q^i}, \tag{5.2}$$

where

$$\begin{bmatrix} r \\ i \end{bmatrix} = \frac{F_r}{F_i L_{r-i}^{q^i}}, \quad \begin{bmatrix} r \\ 0 \end{bmatrix} = \frac{F_r}{L_r}, \quad \begin{bmatrix} r \\ r \end{bmatrix} = 1, \tag{5.3}$$

and

$$\begin{aligned} F_r &= [r][r-1]^q \dots [1]^{q^{r-1}}, \quad && F_0 = 1, \\ L_r &= [r][r-1] \dots [1], && L_0 = 1, \\ [r] &= x^{q^r} - x. \end{aligned} \tag{5.4}$$

Let K be any extension field of $\mathrm{GF}(q, x)$. By (5.2), the functions associated to the polynomials $\psi_r(t)$ are linear operators on the $\mathrm{GF}(q)$-vector space K. Furthermore, $\psi_r(x^r) = \psi_r(m) = F_r$, for m monic of degree r, so that F_r is the product of all monic polynomials in $\mathrm{GF}[q, x]$ of degree r. On the other hand, L_r may be seen to be the l.c.m. of all polynomials in $\mathrm{GF}[q, x]$ of degree r [2].

Following Carlitz [4], we define $g_k \in \mathrm{GF}[q, x]$, and polynomials $G_k(t)$, $G_k^*(t)$ over $\mathrm{GF}[q, x]$. Let k be a positive integer, and write

$$k = e_0 + e_1 q + \dots + e_s q^s \quad (0 \leqslant e_i < q). \tag{5.5}$$

Define g_k by

$$g_k = F_1^{e_1} \dots F_s^{e_s}, \quad g_0 = 1, \tag{5.6}$$

and $G_k(t)$ and $G_k^*(t)$ by

$$G_k(t) = \psi_0^{e_0}(t) \dots \psi_s^{e_s}(t), \quad G_0(t) = 1 \tag{5.7}$$

and

$$G_k^*(t) = \prod_{i=0}^{s} G_{e_i q^i}^*(t), \tag{5.8}$$

where

$$G_{eq^i}^*(t) = \begin{cases} \psi_i^e(t) & \text{for} \quad 0 \leqslant e < q-1, \\ \psi_i^e(t) - F_i^e & \text{for} \quad e = q-1. \end{cases} \tag{5.9}$$

Let K be any extension field of $\mathrm{GF}(q, x)$. Since $\deg G_n(t) = \deg G_n^*(t) = n$, the sequences $\{G_n(t)/g_n\}$ and $\{G_n^*(t)/g_n\}$ are ordered bases of the K-vector space $K[t]$. Indeed, for any $f(t) \in K[t]$ of degree $\leqslant n$, we have [4] the unique representations

$$f(t) = \sum_{i=0}^{n} A_i \frac{G_i(t)}{g_i} \tag{5.10}$$

and

$$f(t) = \sum_{i=0}^{n} A_i^* \frac{G_i^*(t)}{g_i}, \tag{5.11}$$

where A_i is uniquely determined by choosing any r such that $i < q^r$, and setting

$$(5.12) \qquad A_i = (-1)^r \sum_{\deg m < r} \frac{G^*_{q^r-1-i}(m)}{g_{q^r-1-i}} f(m) \qquad (m \in \mathrm{GF}[q, x]),$$

and A^*_i is uniquely determined by choosing any r such that $n < q^r$, and setting

$$(5.13) \qquad A^*_i = (-1)^r \sum_{\deg m < r} \frac{G_{q^r-1-i}(m)}{g_{q^r-1-i}} f(m) \qquad (m \in \mathrm{GF}[q, x]).$$

Note the difference between the defining conditions for r in (5.12) and (5.13).

An important property of the polynomials $G_i(t)/g_i$ and $G^*_i(t)/g_i$ is the fact that for all $m \in \mathrm{GF}[q, x]$, $G_i(m)/g_i \in \mathrm{GF}[q, x]$ and $G^*_i(m)/g_i \in \mathrm{GF}[q, x]$ [4]. With (5.12) and (5.13), this implies that $\{G_i(t)/g_i\}$ and $\{G^*_i(t)/g_i\}$ are, in fact, ordered bases of the $\mathrm{GF}[q, x]$-module of polynomials over $\mathrm{GF}(q, x)$ that map $\mathrm{GF}[q, x]$ into itself.

Moreover, since $\mathrm{GF}[q, x]$ is dense in I_π and the polynomials $G_i(t)/g_i$ and $G^*_i(t)/g_i$ are, by an earlier observation, continuous functions, it follows that $a \in I_\pi$ implies that $G_i(a)/g_i$ and $G^*_i(a)/g_i \in I_\pi$. With (5.12) and (5.13) this implies that $\{G_i(t)/g_i\}$ and $\{G^*_i(t)/g_i\}$ are ordered bases of the I_π-module of polynomials over F_π that map I_π into itself.

Hence, by Theorem 4.5, for every continuous function $f\colon I_\pi \to I_\pi$, there exist null sequences $\{B_i\}$ and $\{B^*_i\}$ in I_π such that

$$(5.14) \qquad f(t) = \sum_{i=0}^{\infty} B_i \frac{G_i(t)}{g_i}$$

and

$$(5.15) \qquad f(t) = \sum_{i=0}^{\infty} B^*_i \frac{G^*_i(t)}{g_i},$$

where (5.14) and (5.15) converge uniformly on I_π.

The coefficients B_i in (5.14) are uniquely determined by f. For if n is any positive integer, the finite sum

$$(5.16) \qquad \sum_{i=0}^{q^n-1} B_i \frac{G_i(t)}{g_i}$$

is the unique polynomial of degree $\leqslant q^n - 1$ which takes the same values as f on the set of all polynomials in $\mathrm{GF}[q, x]$ of degree $< n$. Hence, by (5.12),

$$(5.17) \qquad B_i = (-1)^r \sum_{\deg m < r} \frac{G^*_{q^r-1-i}(m)}{g_{q^r-1-i}} f(m) \qquad (i < q^r).$$

The question of the unconditional uniqueness of the coefficients B_i^* remains open.

Interpolation series of the type which appears in (5.14) may be used to characterize continuous linear operators on the GF(q)-vector space I_π.

THEOREM 5.1. *Let $f\colon I_\pi \to I_\pi$ be continuous. If the (unique) interpolation series for f constructed from the Carlitz polynomials is given by*

$$f(t) = \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i}, \tag{5.18}$$

then f is a linear operator on the GF(q)*-vector space I_π if and only if $A_i = 0$ for $i \neq q^k$, where $k \geqslant 0$.*

Proof. Sufficiency. If $A_i = 0$ for $i \neq q^k$, where $k \geqslant 0$, (5.18) becomes

$$f(t) = \sum_{k=0}^{\infty} A_{q^k} \frac{\psi_k(t)}{F_k}. \tag{5.19}$$

Since, by (5.2), the partial sums of (5.19) are linear operators, it follows immediately that f is a linear operator.

Necessity. We require the following identities [4]:

$$G_i(\lambda t) = \lambda^i G_i(t) \qquad (\lambda \in \mathrm{GF}(q)), \tag{5.20}$$

$$G_i(t_1+t_2) = \sum_{j=0}^{i} \binom{i}{j} G_j(t_1) G_{i-j}(t_2). \tag{5.21}$$

Let $\lambda \in \mathrm{GF}(q)$ be a primitive root of unity. Then (5.18), (5.20), and $f(\lambda t) = \lambda f(t)$, yield

$$\sum_{i=0}^{\infty} \lambda A_i \frac{G_i(t)}{g_i} = \sum_{i=0}^{\infty} \lambda^i A_i \frac{G_i(t)}{g_i}, \tag{5.22}$$

and so $A_i = 0$, unless $i \equiv 1 \pmod{q-1}$.

From (5.18), (5.21), and $f(t_1+t_2) = f(t_1)+f(t_2)$, we infer that

$$\sum_{i=0}^{\infty} A_i \frac{G_i(t_1)}{g_i} + \sum_{i=0}^{\infty} A_i \frac{G_i(t_2)}{g_i} = \sum_{i=0}^{\infty} \frac{G_i(t_1)}{g_i} \sum_{j=i}^{\infty} \frac{g_i}{g_j} \binom{j}{i} A_j G_{j-i}(t_2). \tag{5.23}$$

Equating coefficients of $G_0(t_1)$, we see that $A_0 = 0$. Equating coefficients of $G_i(t_1)/g_i$ for $i > 0$, and subtracting A_i, we get

$$\sum_{j=i+1}^{\infty} \frac{g_i}{g_j} \binom{j}{i} A_j G_{j-i}(t_2) = 0. \tag{5.24}$$

Hence, for all i, j with $1 \leqslant i < j$,

$$\binom{i}{j} A_j = 0. \tag{5.25}$$

It follows that $A_j = 0$ unless $j = p^t$, where p is the characteristic of $GF(q)$. Since $p^t \equiv 1 (\mathrm{mod}\, q-1)$, we must have $p^t = q^k$, where $k \geqslant 0$.

References

[1] Y. Amice, *Interpolation p-adique*, Bull. Soc. Math. France 92 (1964), pp. 117–180.

[2] L. Carlitz, *On polynomials in a Galois Field*, Bull. Amer. Math. Soc. 38 (1932), pp. 736–744.

[3] — *On certain functions connected with polynomials in a Galois Field*, Duke Math. J. 1 (1935), pp. 137–168.

[4] — *A set of polynomials*, Duke Math. J. 6 (1940), pp. 486–504.

[5] — *Finite sums and interpolation formulas over* $GF[p^n, x]$, Duke Math. J. 15 (1948), pp. 1001–1012.

[6] J. Cassels and A. Fröhlich, *Algebraic Number Theory*, Washington, D.C., 1957.

[7] J. Dieudonné, *Sur les fonctions continues p-adiques*, Bull. Sci. Math. 68 (1944), pp. 79–95.

[8] K. Mahler, *An interpolation series for a continuous function of a p-adic variable*, J. Reine Angew. Math. 199 (1958), pp. 23–34.

[9] W. Schöbe, *Beiträge Zur. Funktion theorie in nichtarchimedisch bewerteten Körpern*, Munster 1930.

[10] A. Weil, *Basic Number Theory*, New York 1967.

DUKE UNIVERSITY
THE UNIVERSITY OF TENNESSEE

Received on 27. 2. 1970