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A functional equation arising in multi-agent statistical decision theory

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Dedicated to Professor Otto Haupt with best wishes on his 100th birthday.

Abstract. Using recent results of Járai we show that the measurable solutions of the functional equation $f(x_1y_1,\ldots,x_ny_n)f((1-x_1)(1-y_1),\ldots,(1-x_n)(1-y_n)) = f(x_1(1-y_1),\ldots,x_n(1-y_n))f(y_1(1-x_1),\ldots,y_n(1-x_n))$, where $f\colon (0,1)^n \to (0,\infty)$ and $0 < x_i, y_i < 1$, $i = 1,\ldots,n$, are of the form

$$f(x_1,\ldots,x_n) = c\exp\left(\sum_{i=1}^{n} a_i(x_i - x_i^2)\right)\prod_{i=1}^{n} x_i^{b_i},$$

where $c > 0$, $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ are arbitrary real constants. This result enables one to characterize certain independence-preserving methods of aggregating probability distributions over four alternatives.

When the probability distributions of several experts are to be aggregated into a single "consensual" distribution, it is often recommended that this be done in such a way as to preserve any agreement regarding the independence of various events. For probability spaces containing at least five points, however, Genest and Wagner [to appear] have shown that, subject to a mild additional restriction, the only independence-preserving aggregation methods are those which endorse the opinion of a single expert. When just four alternatives are present, on the other hand, the same conditions allow a rich variety of non-dictatorial aggregation methods. Indeed, followed by a suitable normalization, any function $f\colon(0,1)^n \to (0,\infty)$ satisfying the functional equation.

$$f(x_1y_1,\ldots,x_ny_n)f((1-x_1)(1-y_1),\ldots,(1-x_n)(1-y_n)) = f(x_1(1-y_1),\ldots,x_n(1-y_n))f(y_1(1-x_1),\ldots,y_n(1-x_n)), \quad (1)$$

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for $0 < x_i, y_i < 1$, $i = 1, \ldots, n$, yields an independence-preserving method for combining the probability assignments of n experts.

The measurable solutions of (1) are of the form

$$f(x_1, \ldots, x_n) = c \exp\left(\sum_{i=1}^{n} a_i(x_i - x_1^2)\right) \sum_{i=1}^{n} x_i^{b_i}, \tag{2}$$

where $c > 0$, $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real constants, a result established by Abou-Zaid [1984].

In what follows the Roman capital letters, R, S, T, U, V, X, and Y denote n-dimensional vectors, $\underline{0}$ and $\underline{1}$ the n-dimensional vectors $(0, \ldots, 0)$ and $(1, \ldots, 1)$, respectively, and all operations and relations are taken coordinatewise. With this convention we may, for example, abbreviate formula (1) by

$$f(\mathrm{XY})f((\underline{1} - \mathrm{X})(\underline{1} - \mathrm{Y})) = f(\mathrm{X}(\underline{1} - \mathrm{Y}))f(\mathrm{Y}(\underline{1} - \mathrm{X})),$$

where $\underline{0} < \mathrm{X}, \mathrm{Y} < \underline{1}$.

Abou-Zaid's solution of the above equation involved the solution of a much more general functional equation by a painstaking induction on n, the cases $n = 1,2$ having been treated by Kannappan and Ng [1980]. At the Twenty-third International Symposium on Functional Equations (Gargnano [1985]) Wagner posed the problem of deriving (2) directly from (1) by an argument of reasonable length. In response, Járai suggested that the following result might provide the first step of such a derivation:

LEMMA (Járai [1985]), *Every measurable solution of* (1) *is infinitely differentiable.*

Proof. Letting $T = \mathrm{XY}$, the vector abbreviation of (1) becomes

$$f(\mathrm{T}) = f(\mathrm{Y} - \mathrm{T}) + f((\mathrm{T}/\mathrm{Y}) - \mathrm{T}) - f(\underline{1} + \mathrm{T} - \mathrm{Y} - (\mathrm{T}/\mathrm{Y})),$$

with $\underline{0} < \mathrm{T} < \mathrm{Y} < \underline{1}$. Hence f satisfies the hypotheses of Theorem 1.3 (Járai [1986]), which imply that f is infinitely differentiable.

In this note we use Járai's Lemma to offer a new proof of the

THEOREM. *The measurable solutions of* (1) *are given by* (2).

Proof. Let $F = \log f$, where f satisfies (1). Then

$$F(\mathrm{RS}) + F((\underline{1} - \mathrm{R})(\underline{1} - \mathrm{S})) = F(\mathrm{R}(\underline{1} - \mathrm{S})) + F(\mathrm{S}(1 - \mathrm{R})), \tag{3}$$

where $\underline{0} < \mathrm{R,S} < \underline{1}$. Letting $\mathrm{U} = \mathrm{RS}$, $\mathrm{V} = \mathrm{R}(\underline{1} - \mathrm{S})$, $\mathrm{X} = (\underline{1} - \mathrm{R})\mathrm{S}$ and $\mathrm{Y} = (\underline{1} - \mathrm{R})(\underline{1} - \mathrm{S})$, (3) becomes

$$F(\mathrm{U}) - F(\mathrm{V}) = F(\mathrm{X}) - F(\mathrm{Y}), \tag{4}$$

where $\mathrm{U, V, X, Y} > \underline{0}$, $\mathrm{U} + \mathrm{V} < \underline{1}$, and $\mathrm{X} + \mathrm{Y} < \underline{1}$. We note for future reference that U, V, X, and Y are related by the formulas

$$\mathrm{U} = \frac{\mathrm{X}}{\mathrm{X}+\mathrm{Y}} - \mathrm{X} \qquad \mathrm{X} = \frac{\mathrm{U}}{\mathrm{U}+\mathrm{V}} - \mathrm{U} \tag{5}$$

$$\mathrm{V} = \frac{\mathrm{Y}}{\mathrm{X}+\mathrm{Y}} - \mathrm{Y} \qquad \mathrm{Y} = \frac{\mathrm{V}}{\mathrm{U}+\mathrm{V}} - \mathrm{V}.$$

It follows from (4) that

$$\partial^2/\partial x_i \partial y_j (F(\mathrm{U}) - F(\mathrm{V})) = 0, \ \mathrm{i, j} = 1, \ldots, \mathrm{n},$$

the requisite differentiability of F following from Járai's Lemma. Expanding the above (with F_i and F_{ij} denoting, respectively, the partial derivative with respect to the i-th variable and the second partial with respect to the i-th and j-th variables) we obtain

$$F_{ij}(\mathrm{U}) \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial y_j} + F_j(\mathrm{U}) \frac{\partial^2 \mathrm{u}_j}{\partial x_i \partial y_j} \tag{6}$$

$$- F_{ij}(\mathrm{V}) \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial y_j} - F_j(\mathrm{V}) \frac{\partial^2 v_j}{\partial x_i \partial y_j} = 0.$$

We first treat the case $n = 1$, for which equation (6) becomes

$$F''(u) \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + F'(u) \frac{\partial^2 u}{\partial x \partial y} - F''(v) \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} - F'(\mathrm{V}) \frac{\partial^2 v}{\partial x \partial y} = 0. \tag{7}$$

Computations based on (5) establish that

$$\left.\begin{aligned}
\partial u/\partial x &= [v/(u+v)(1-u-v)] - 1 \\
\partial u/\partial y &= -u/(u+v)(1-u-v) \\
\partial^2 u/\partial x \, \partial y &= (u-v)/(u+v)(1-u-v)^2 \\
\partial v/\partial x &= -v/(u+v)(1-u-v) \\
\partial v/\partial y &= [u/u+v)(1-u-v)] - 1 \\
\partial^2 v/\partial x \, \partial y &= (v-u)/(u+v)(1-u-v)^2,
\end{aligned}\right\} \tag{8}$$

and (7) and (8) yield

$$F''(u)[u - (u + v)^2]u + F'(u)[u^2 - v^2] = F''(v)[v - (u + v)^2] + F'(v)[v^2 - u^2] \quad (9)$$

Setting $G(u) = uF'(u)$, equation (9) becomes

$$G'(u)[u - (u + v)^2] + G(u)[2u + 2v - 1] = G'(v)[v - (u + v)^2] + F(v)[2u + 2v - 1]. \quad (10)$$

valid for all u, v such that $u > 0$, $v > 0$, and $u + v < 1$. For fixed small positive v, (10) is a first order linear differential equation for $G(u)$. Multiplying (10) by $[u - (u + v)^2]^{-2}$ and integrating with respect to u yields

$$\frac{G(u)}{u - (u + v)^2} = \frac{-(u + v)G'(v)}{u - (u + v)^2} + \frac{G(v)}{u - (u + v)^2} + C(v). \quad (11)$$

Setting $u = v$ (assuming $v < 1/2$, so that $u + v < 1$) in (11) and solving for $C(v)$ yields

$$C(v) = 2G'(v)/(1 - 4v),$$

and substituting this expression in (11) and simplifying yields

$$G(u) = \frac{G'(v)[u - 2u^2]}{1 - 4v} - \frac{G'(v)[v - 2v^2]}{1 - 4v} + G(v)$$

for $0 < u < 1 - v$. Since v may be chosen arbitrarily small positive, G must take the form

$$G(u) = a(u - 2u^2) + b$$

for some constants a and b. Since $G(u) = uF'(u)$, F must therefore take the form

$$F(u) = a(u - u^2) + b \log u + c \quad (12)$$

for constants a, b, and c, which yields (2) for $f = \exp(F)$ when $n = 1$.

If $n > 1$, consider equation (6) for $i = j$. The above argument for $n = 1$ then shows that

$$F(\mathrm{U}) = a_i(u_1, \ldots, \hat{u}_i, \ldots, u_n)(u_i - u_i^2) + b_i(u_1, \ldots, \hat{u}_i, \ldots, u_n)\log u_i + c_i(u_1, \ldots, \hat{u}_i, \ldots, u_n)$$

for $i = 1, \ldots, n$, where "$\hat{\ }$" indicates omission of the corresponding variable. Thus F must have the form

$$F(\mathrm{U}) = \sum_{1 \leqq s_1, \ldots, s_n \leqq 3} A_{s_1, \ldots, s_n} \varphi_{s_1}(u_1) \ldots \varphi_{s_n}(u_n), \tag{13}$$

where $\varphi_1(u) = u - u^2$, $\varphi_2(u) = \log u$, $\varphi_3(u) = 1$, and the $A_{s_1, \ldots, s_n}$ are constants.

Now equation (6) for $i \neq j$ reads

$$F_{ij}(\mathrm{U}) \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial y_j} = F_{ij}(\mathrm{V}) \frac{\partial v_i}{\partial x_i} \frac{\partial v_j}{\partial y_j}. \tag{14}$$

From (13),

$$F_{ij}(\mathrm{U}) = \sum_{1 \leqq s_1, \ldots, s_n \leqq 3} A_{s_1, \ldots, s_n} \varphi_{s_1}(u_1) \ldots \varphi'_{s_i}(u_i) \ldots \varphi'_{s_j}(u_j) \ldots \varphi_{s_n}(u_n). \tag{15}$$

Combining (14) and (8) we get, upon simplification,

$$F_{ij}(\mathrm{U})[u_i - (u_i + v_i)^2]u_j = F_{ij}(\mathrm{V})[v_j - (u_j + v_j)^2]v_i. \tag{16}$$

If $n > 2$ and $k \neq i, j$, it is clear from (16) that $F_{ij}(\mathrm{U})$ does not depend on u_k, and hence from (15) that

$$F_{ij}(\mathrm{U}) = \sum_{1 \leqq s_i, s_j \leqq 3} A_{s_i, s_j} \varphi'_{s_i}(u_i) \varphi'_{s_j}(u_j) \tag{17}$$

for constants A_{s_i, s_j}. Substituting (17) into (16) yields

$$u_j \sum_{1 \leqq s_i, s_j \leqq 3} A_{s_i, s_j} \varphi'_{s_i}(u_i) \varphi'_{s_j}(u_j)[u_i - (u_i + v_i)^2]$$

$$= v_i \sum_{1 \leqq s_i, s_j \leqq 3} A_{s_i, s_j} \varphi'_{s_i}(v_i) \varphi'_{s_j}(v_j)[v_j - (u_j + v_j)^2]. \tag{18}$$

Multiplying each side of (18) by $u_i u_j v_i v_j$ and simplifying by cancellation yields

$$v_j P(u_i, u_j)[u_i - (u_i + v_i)^2] = u_i P(v_i, v_j)[v_j - (u_j + v_j)^2], \tag{19}$$

where P is a polynomial. The irreducible polynomial $u_i - (u_i + v_i)^2$ must then divide the right-hand side of (19), which clearly cannot happen unless P is identically zero. Hence both sides of (19) are identically zero, and so $F_{ij}(\mathrm{U})$ is identically zero, which implies that in each term of (13) there is at most one nonconstant factor. Thus

$$F(\mathrm{U}) = \sum_{i=1}^{n} a_i(u_1 - u_i^2) + \sum_{i=1}^{n} b_i \log u_i + c,$$

for constants (a_i), (b_i), and c, from which it follows that $f = \exp(F)$ is given by (2), as asserted.

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