A NOTE ON SOME CONFUSING FUNCTIONAL NOTATION Carl Wagner (Feb. 4, 2013)

It is standard practice, and unambiguous, to express the statement “ f is a function from the domain A to the co-domain B” symbolically by: $f : A \to B$. If f is bijective, then for each $b \in B,$ $f^{-1}(b)$ denotes the unique $a \in A$ such that $f(a) = b.$ Note that $f^{-1} : B \to A$, and

(1) for all $a \in A,$ $f^{-1}(f(a)) = a$ and

(2) for all $b \in B,$ $f(f^{-1}(b)) = b.$

The function f^{-1} is called the *inverse* of f.

Somewhat confusingly, the symbol f is also used to denote a function from 2^A to 2^B. So construed, f is defined for all $E \subset A$ by

(3) $f(E) := \{f(a) : a \in E\}.$

Even more confusingly, for *every* function $f : A \to B$, **even if** f **is not bijective**, the symbol f^{-1} is used to denote a function from 2^B to 2^A, defined for all $H \subset B$ by

(4) $f^{-1}(H) := \{a \in A : f(a) \in H\}.$

The set $f^{-1}(H)$ is called the *pre-image* of H.

Are there analogues of (1) and (2) when f and f^{-1} are defined by (3) and (4)? As an exercise, try the fill in the blanks in the following statements with the appropriate symbol ($=, \subset,$ or $\supset$):

(5) For all $E \subset A,$ $f^{-1}(f(E))$ ______ $E.$

(6) For all $H \subset B,$ $f(f^{-1}(H))$ ______ $H.$

If you fill in either (5) or (6) with $\subset$ or $\supset,$ are there additional conditions on f that would enable you to substitute the symbol $"="$ for the aforementioned symbols?

The symbol $f^{\leftarrow}$. This symbol occurs less frequently in mathematical writing, and denotes a function from the set B (**not** 2^B) to the set 2^A, defined for all $b \in B$ by

(7) $f^{\leftarrow}(b) := \{a \in A : f(a) = b\}.$ In other words, $f^{\leftarrow}(b) = f^{-1}(\{b\})$.