that will not be easy, because these relationships are themselves not easy: they involve serious social inequalities, some more obvious and others, with the least obvious ones being perhaps the most important ones to understand and respond to.

Love as a Guide to Morals does tell us that we ought to practice mindful attentiveness to the Other. It suggests quiet meditation as way to focus less on the self and more on the self's relationships to other selves. And there is something in this. I share the book's belief that competitive individualism as a guide to morals robs us of our souls. But what is not to be found in quiet meditation is the hard and nasty work of confronting morally illegitimate power inequalities as they actually manifest themselves in our lives together with one another. No amount of meditation will enable me to understand how my position of social power may be experienced as deeply oppressive to those I have power over. This requires conversation, difficult conversation, conversation of the fracturing kind that has occurred within feminist groups where, for example, women of color have confronted white women over their own illegitimate forms of power as white women in profoundly white societies.

These conversations are not easy. It is hard to find social locations in which to have them. It is hard not to let them lead us into even more deeply fractured relationships. But this is the necessary and very real hard work of morality, whether we call it phronesis or praxis. But however we think about the practice of the moral life, Aristotelian phronesis is not the socially aware kind of praxis explored by contemporary approaches to feminist ethics, and so for me and my classroom, virtuous love is not enough of a basis to guide the exploration of our shared morals.

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An Introduction to the Philosophy of Mathematics

Mark Colyvan

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CARL WAGNER

I wish that this book had been available when I was an undergraduate. As a mathematics major with an interest in philosophy, I was keen to learn about the philosophy of mathematics. But what I was able to find was not particularly exciting: all the 5 + 7 = 12 business, with discussions of whether mathematical theorems were synthetic a priori, or simply analytic, propositions, mostly written as if mathematics consisted only of elementary number theory and geometry. Somewhat more interesting, but also puzzling, were accounts of the logicist, formalist and intuitionist foundations of mathematics. Could David Hilbert, who

pp. 316–320
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made major contributions to mathematic physics, *really* have asserted that mathematics was nothing more than a symbolic game? And how could intuitionists deny the law of excluded middle, since every declarative sentence was surely either true or false (quite apart from whether it was possible to determine which)? I was fortunate to be able to take courses in logic from Alonzo Church and Paul Benacerraf, but they concentrated on the formal development of the subject, with little attention to philosophical issues. Benacerraf kindly gave me a draft of his seminal article, "What Numbers Could Not Be," which appeared in 1965, and I read it with great interest and the sense of encountering something new and exciting. But by then I was headed off to graduate school in mathematics, and pretty much abandoned further engagement with the philosophy of mathematics.

At the very beginning of *An Introduction to the Philosophy of Mathematics* I found an elucidation of my undergraduate malaise:

> The first half of the twentieth century was a golden age for philosophy of mathematics. . . . Sadly, the excitement of these times didn't last. The debates over the foundations of mathematics bogged down. After a very productive 30 or 40 years, very little progress was made thereafter, and, by and large, both philosophers and mathematicians became tired of the philosophy of mathematics.

The author continues:

> It is very easy, as a student of philosophy of mathematics, to spend one's time looking back to the debates and developments of the first half of the twentieth century. But the philosophy of mathematics has moved on, and it is once again engaged with mathematical practice. (2–3)

Accordingly, Colyvan deals with what he calls the "big isms," namely, formalism, logicism, and intuitionism, in just a few pages of his first chapter. In a masterly piece of compressed exposition, he gives a clear account of these very different approaches to furnishing foundations for mathematics (and clears up my undergraduate confusion about the intuitionist view of the law of excluded middle by distinguishing between its semantic and syntactic versions). He then outlines how the agenda for contemporary philosophy of mathematics was shaped by two papers by Paul Benacerraf. The first of these, mentioned above, pointed out the problem of underdetermination in set-theoretic constructions of the natural numbers, i.e., the fact that there are different ways of effecting such constructions. The second, "Mathematical Truth," which appeared in 1973, posed two additional problems: (i) provide a semantics that is uniform across mathematical and non-mathematical discourse, and (ii) provide a plausible epistemology for mathematics. The difficulty is that if we respond to the first of these problems with the classical Aristotelian answer, "to say of what is that it is is true," etc., we are on the road to being Platonists, believing that numbers have an independent, abstract existence. But if we subscribe to a causal theory of knowledge, it is hard to see how to account for our presumed knowledge about things which are abstract entities. These issues are elucidated in more detail in the remainder of the book, which also contains a number of additional features unique to such an introductory text.

Chapter 2 begins with a discussion of the Skolem-Löwenheim theorem ("If a first order theory has a model, then it has a countable model."), and a proof of Cantor's theorem ("The power set of any set S has cardinality strictly greater than that of S."), and dispels the apparent contradiction between these two results. Also discussed are Gödel's incompleteness theorems (but not Church's theorem), the Gödel-Cohen theorem on the independence of the continuum hypothesis, and realist (i.e., Platonist) and anti-realist responses to the latter.

Chapter 3 delves more deeply into questions about the existence of mathematical objects, elaborating an earlier discussion of mathematical realism, and introducing structuralism, the view that mathematics is about relations, the nature of the objects so related being immaterial. There is also a detailed account of the Quine-Putnam indispensability argument for mathematical realism ("We ought to have an ontological commitment to all and only the entities that are indispensable to our current best scientific theories. Mathematical entities are indispensable to our best scientific theories. Ergo . . ."), and various objections to this argument from Hartry Field, Penelope Maddy, and Elliot Sober. Colyvan has a real talent for conveying the excitement of these ongoing debates, and encouraging readers to develop their own views on these issues by means of well-chosen discussion questions at the end of this (and every other) chapter. There is a maxim of mathematical pedagogy to the effect that "mathematics is not a spectator sport," and Colyvan has succeeded in ensuring that the same is true of the philosophy of mathematics.

Chapter 4 continues in this vein with an account of various sorts of nominalism, including fictionalism, in which there are no abstract mathematical objects, and mathematical truths are simply parts of certain mathematical "stories."

Chapter 5, on mathematical explanation, stands out even from the other uniformly excellent chapters of this book. The author proposes to take seriously the possibility of what he calls "intra-mathematical explanation," i.e., of proofs in mathematics that are not merely guarantees of truth, but genuinely enlightening regarding the connection between the hypotheses and conclusions of the theorems being proved. While proofs by contradiction, or by mathematical induction, are often regarded as failing to be explanatory, Colyvan suggests that a more extensive survey of proofs of this type, especially those involving transfinite induction, may well turn up exceptions to this view. (I would add here that explanatoriness need not be thought of as an all-or-nothing property. Indeed, one can typically boost the explanatoriness of a proof of B from hypotheses $A_1,\ldots,A_n$ by exhibiting, for each $i = 1,\ldots n$, an example for which it is true that $A_1\wedge\ldots\wedge A_{i-1}\wedge\neg A_i\wedge A_{i+1}\wedge\ldots\wedge A_n\wedge\neg B$, thereby showing that each hypothesis is essential. Indeed, it is good mathematical pedagogy—often, sadly, ignored by mathematics teachers—to supplement proofs with such examples, or to assign as exercises the construction of such examples.) The author also points to the phenomenon of generalization in mathematics, both in the sense of extending a given mathematical structure (as in the case of embedding the natural numbers in the field of complex numbers and extending exponentiation to this larger domain), and in the sense of abstracting away from a concrete structure (as in the case of observing that certain theorems of elementary number theory actually hold in any group), and argues convincingly that such generalizations have explanatory power.

The discussion of extra-mathematical explanation in the latter part of chapter 5 is particularly interesting. Countering the view that the mathematics employed by scientists is merely a descriptive tool that carries no explanatory load, Colyvan provides several examples in which mathematics plays an essential role in explaining a given phenomenon. These include intriguing explanations of why the life cycles of certain North American cicadas are prime numbers, why hive-bee honeycomb has a hexagonal structure, and why there are relatively few asteroids in certain regions of the asteroid belt between Mars and Jupiter. The chapter concludes with an admirably clear explanation of the so-called Lorentz length contraction, in which the length of a body in motion relative to some observer, as measured by this observer, is seen to decrease in the direction of the motion. Here, in contrast to proffered mechanical explanations for this phenomenon, it is now accepted that the preferred explanation is given by the geometry of Minkowski space.

Chapter 6 deals with the philosophy of applied mathematics, including the problem of accounting for the "unreasonable effectiveness of mathematics" in scientific applications. The material in this chapter is too rich to be covered in a short review. But one point made by the author is particularly worth noting, namely, that mathematical models of physical phenomena may approximate these phenomena by means of structures of *greater* complexity than that of the phenomena in question. Indeed, this is the case whenever one employs the real number system and the differential and integral calculus (rather than the finite difference calculus) to study what are, insofar as we can know them, discrete systems.

Chapter 7 ("Who's Afraid of Inconsistent Mathematics?") will set the teeth of many mathematicians on edge, but it is a delightful exploration of the seemingly outrageous suggestion that inconsistency in a mathematical theory need not render that theory useless. The author begins by pointing to naïve set theory and the early calculus of infinitesimals as examples of inconsistent theories in which mathematicians worked productively for over a hundred years. A mathematician would reply to this that, by following certain rules of thumb, earlier mathematicians managed to operate within a consistent fragment, never fully articulated, of these theories, but that being sanguine about allowing propositions of the form $P \wedge \neg P$ into our corpus of theorems can only spell disaster. This is because, simply on the basis of (two-valued) propositional logic, the inference from $P \wedge \neg P$ to Q, where Q is any proposition whatsoever, is valid. Here is where things get interesting (and a little wild, for the conservatives among us). Suppose that our criterion for validity were based on the three truth values $\{1,i,0\}$, where 1 designates classical truth, 0 designates classical falsehood, and i designates a third value, which Colyvan suggests interpreting as "true and false." The truth tables for the connectives $\neg$, $\wedge$, and $\vee$ are gotten by extending the classical tables by means of the following stipulations:

(i) $\neg i=i$;

(ii) $i\wedge 1=1\wedge i=i\wedge i=i$ and $i\wedge 0=0\wedge i=0$; and

(iii) $i\vee 1=1\vee i=1$ (*not i*, as the typos in the book have it) and $i\vee 0=0\vee i=i\vee i=i$.

Terming the values 1 and *i* "designated," and the value 0 "non-designated," one then defines an argument form as valid-according-to-the-logic-of paradox ("LP-valid") if, whenever all of the hypotheses of the argument take designated values, the conclusion is guaranteed to take a designated value. LP is a conservative extension of classical propositional logic (no new valid arguments arise), but certain classically valid arguments are not LP-valid. In particular, the argument from $P\wedge\neg P$ to Q fails to be LP-valid (let P have the value i and Q the value 0). It isn't clear how comforting this result is, however. As Colyvan indicates in one of the exercises, modus ponens fails to be LP-valid. Furthermore, proof by contradiction is unsupported in LP, since the inference from $Q\supset(P\wedge\neg P)$ to $\neg Q$ is not LP-valid (let P have the value i and Q the value 1). So the rules of inference of LP are severely impoverished. Nevertheless, LP (one of several so-called "paraconsistent" logics) is great fun to play with, and furnishes some delightful exercises for students to work through.

Chapter 8 highlights the importance of notation in enhancing the salience (and, indeed, discovery) of mathematical results, as well as of the delicate business of choosing definitions, with a nice summary of the observations of Lakatos (in *Proofs and Refutations*) on the evolution of the definition of the term "polyhedron" in connection with Euler's formula.

Chapter 9 ("Epilogue: Desert Island Theorems") contains a list and short discussion of some mathematical theorems of particular interest to philosophers, as well as some of the author's personal favorites, and a list of some famous open problems in mathematics. The only improvements on this otherwise excellent chapter that I would suggest are (i) that Bayes's Theorem should also be presented in the beautifully simple "odds form"

$$\frac{P(H \mid E)}{P(\neg H \mid E)} = \frac{P(H)}{P(\neg H)} \text{ x } \frac{P(E \mid H)}{P(E \mid \neg H)}$$

("posterior odds = prior odds x likelihood ratio"), and (ii) that the author's version of the prime number theorem, $\pi(n) \sim \int_2^n dx / \ln x$, where $\pi(n)$ denotes the number of primes less than or equal to n, be supplemented by the less exact, but more salient asymptotic formula $\pi(n) \sim n / \ln n$, which has a nice probabilistic interpretation ("the probability that a number chosen randomly from the set {1,…,n} is prime is approximately $1 / \ln n$").

This book, while perhaps written primarily for philosophy students, could also be very profitably read by students and teachers of mathematics. Indeed, this reviewer hopes to use it both in a capstone course for undergraduate mathematics majors, and in a graduate seminar for secondary school mathematics teachers.

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