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ANSCOMBE'S PARADOX AND THE RULE OF THREE-FOURTHS

1. ANSCOMBE'S PARADOX

That the resolution of a set of proposals by majority rule may result in outcomes with which a majority of voters disagree in a majority of cases seems first to have been noted by Anscombe (1976). The following example, due to Gorman (1978), illustrates this possibility:

(1.1) *Proposals*

		1	2	3
Voters	1	yes	yes	no
	2	no	no	no
	3	no	yes	yes
	4	yes	no	yes
	5	yes	no	yes

Each of the first three voters disagrees with the outcomes, based on simple majority rule, in a majority of cases, voter 1 with the outcomes on proposals 2 and 3, voter 2 with those on proposals 1 and 3, and voter 3 with those on proposals 1 and 2.

One senses that such situations materialize when issues are decided by a number of 'close' votes. Thus it is natural to ask how substantial, on average, prevailing coalitions[1] must be in order to preclude such situations. We find that when prevailing coalitions comprise, on average, at least three-fourths of those voting, the set of voters disagreeing with a majority of outcomes cannot comprise a majority. In particular, the rule requiring ratification of amendments to the U.S. Constitution by at least three-fourths of the states guarantees that the set of states whose legislatures have rejected a majority of the amendments thus adopted can never constitute a majority.

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2. THE RULE OF THREE-FOURTHS

Suppose that N voters cast yes-or-no votes on a set of K proposals. Given a decision procedure for adopting or rejecting proposals, there corresponds to each $N \times K$ 'voting matrix' an $N \times K$ 'A–D matrix', namely, the matrix the i–jth entry of which is A if voter i agrees with the outcome of the vote on proposal j, as determined by the procedure, and D if he disagrees with that outcome. For example, the A–D matrices

$$(2.1) \qquad M_1 = \begin{bmatrix} A & D & D \\ D & A & D \\ D & D & A \\ A & A & A \\ A & A & A \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} D & D & A \\ A & A & A \\ A & D & D \\ D & A & D \\ D & A & D \end{bmatrix}$$

correspond to the voting matrix (1.1) under the respective decision procedures P_1 (adopt a proposal iff more than half of the voters vote yes) and P_2 (adopt a proposal iff more than two-thirds of the voters vote yes).

Given an $N \times K$ A–D matrix M, whatever the decision procedure by which it arises, if we denote by A_M the number of A's appearing in M, then A_M/K is the average size, across all proposals, of the prevailing coalitions, and A_M/NK the average fraction of voters, across all proposals, comprising the prevailing coalitions.

The following theorem states a condition sufficient to guarantee that the set of voters disagreeing with a majority of the outcomes of voting on a set of proposals cannot constitute a majority:

THEOREM 2.1. *Let M be an $N \times K$ A–D matrix and let A_M denote the number of A's appearing in M. If*

$$(2.2) \qquad A_M > B(N, K) = \left(\left[\frac{N}{2}\right] + 1\right)\left[\frac{K-1}{2}\right] + \left[\frac{N-1}{2}\right]K,$$

where $[x]$ *denotes the greatest integer less than or equal to* x, *then no more than half of the rows of M contain more D's than A's.*

Proof. Suppose that more than half of the rows contained more D's than A's. Then there would be at least $[N/2] + 1$ rows, each containing at most

$[(K-1)/2]$ A's. Since the remaining $N-[N/2]-1=[(N-1)/2]$ rows could contain at most K A's, it would follow that $A_M \leqslant B(N, K)$, contradicting (2.2). ∎

The above theorem may be elaborated as follows:

THEOREM 2.2. *Let*

$$(2.3)\qquad B(N,K) = \left(\left[\frac{N}{2}\right]+1\right)\left[\frac{K-1}{2}\right] + \left[\frac{N-1}{2}\right]K$$

and

$$(2.4)\qquad F(N,K) = B(N,K)/NK.$$

Then

$$(2.5)\qquad F(N,K) < 3/4.$$

for all N, K and

$$(2.6)\qquad \lim_{N,K\to\infty} F(N,K) = 3/4.$$

Proof. By (2.3),

$$(2.7)\qquad \frac{N}{2}\left(\frac{K}{2}-1\right)+\left(\frac{N}{2}-1\right)K \leqslant B(N,K) \leqslant \left(\frac{N}{2}+1\right)\left(\frac{K-1}{2}\right) + \left(\frac{N-1}{2}\right)K < \tfrac{3}{4}NK,$$

and so, by (2.4) and (2.7),

$$(2.8)\qquad \frac{3}{4}-\frac{1}{2K}-\frac{1}{N} < F(N,K) < \frac{3}{4},$$

which yields (2.5) and (2.6). ∎

As a consequence of the above theorems we have the following 'Rule of Three-Fourths': *If N individuals cast yes-or-no votes on K proposals then, whatever the decision method employed to determine the outcomes of the votes on these proposals, if the average fraction of voters, across all proposals, comprising the prevailing coalitions is at least three-fourths, then the set of* *of voters who disagree with a majority of the outcomes cannot comprise a majority.*

We remark that when outcomes are determined by simple majority rule, i.e., when a proposal is adopted iff it receives more yes than no votes, condition (2.2) of Theorem 2.1 is sharp whenever $N \geqslant 4$ and $K \geqslant 3$.[2] For consider the following $N \times K$ voting matrix: Voter 1 votes no on the first $[(K-1)/2]$ proposals, and yes on the remaining proposals. Voter 2 votes yes on the first $[(K-1)/2]$ proposals, no on the next $[(K-1)/2]$ proposals, and yes on the remaining $K-2[(K-1)/2]$ proposals. Voters 3 through $[N/2]+1$ vote no on the last $[(K-1)/2]$ proposals and yes on the remaining proposals. The remaining $[(N-1)/2]$ voters vote no on all proposals. By simple majority rule all proposals are rejected. The first $[N/2]+1$ voters disagree with these outcomes in a majority of cases, and the corresponding A–D matrix actually contains $([N/2]+1)[(K-1)/2]+[(N-1)/2]K = B(N, K)$ A's. Since, by Theorem 2.2, $F(N, K) = B(N, K)/NK \to 3/4$ as $N, K \to \infty$, it follows that when outcomes are decided by majority rule, the rule of three-fourths stated above is the best possible result covering all values of N and K. Indeed, for any $\epsilon < 3/4$ there are an infinite number of voting matrices for which the average size of prevailing conditions is a fraction of those voting exceeding ϵ, and yet a majority of voters disagree with a majority of the outcomes.

3. REQUIRING THE ASSENT OF THREE-FOURTHS

Suppose that a decision procedure required the assent of at least three-fourths of those voting in order to adopt a proposal. Since a prevailing coalition might, in the case of a rejected proposal, comprise only slightly more than a fourth of the voters, the average fraction of voters, across all proposals, comprising the prevailing coalitions might well be less than three-fourths. Thus there may be a majority of voters who disagree with a majority of the outcomes. On the other hand, if we restrict attention only to those proposals *adopted* by the aforementioned procedure, it is clear that those voters disagreeing with a majority of the outcomes of this subset of the full set of proposals cannot comprise a majority. In particular, the rule requiring ratification of amendments to the U.S. Constitution by at least three-fourths (38) of the 50 states guarantees that the set of states whose legislatures have rejected a majority of the amendments actually adopted can never constitute a majority.

As it turns out, the above result would continue to obtain if one required ratification by just 37 States. This follows from Theorem 2.1, since $37K > B(50, K)$. However, ratification by 36 States would not suffice to guarantee this result, as shown by the following (minimal) example:

(3.1)

Amendments	*States Voting Yes*
1–3	1–12, 27–50
4	1–10, 25–50
5	1–9, 11–13, 27–50
6	1, 16–50
7	2–13, 27–50
8	10–21, 27–50
9	13–24, 27–50
10	13–19, 22–50
11	13–17, 20–50
12	14–16, 18–50
13	14–15, 17–50

Each of the 13 amendments is ratified by 36 states, but states 1–26 each vote no on 7 of the amendments.

In general, however, requiring the assent of a fraction of voters less than three-fourths in order to adopt a proposal allows for an infinite number of cases where a majority of voters disagree with a majority of proposals thus adopted. For given any $\epsilon < 3/4$ one can find an infinite number of pairs (N, K) for which there exists an $N \times K$ voting matrix, each column of which contains a fraction of yes votes not less than ϵ, but for which a majority of rows contain a majority of no votes. These matrices may be constructed as follows: Since for each integer r, $3r/(4r + 1) < 3/4$ and $\lim_{r\to\infty} 3r/(4r + 1) = 3/4$, one can find an infinite number of integers n such that $\epsilon \leqslant 3n/(4n + 1) < 3/4$. Let $N = 4n + 1$ and $K = 2n + 1$. It is convenient to label the N voters $0, 1, \ldots, 4n$ and the K proposals $0, 1, \ldots, 2n$. Now define the $N \times K$ voting matrix (v_{ij}), $0 \leqslant i \leqslant 4n$, $0 \leqslant j \leqslant 2n$, as follows: For each $j = 0, 1, \ldots, 2n$, $v_{ij} =$ yes iff $2n + 1 \leqslant i \leqslant 4n$, or $0 \leqslant i \leqslant 2n$ and there exists a k, $0 \leqslant k \leqslant n - 1$, such that $i \equiv j + k \pmod{2n + 1}$. In each column of this matrix there are $3n$ yes votes, and in each of the first $2n + 1$ rows there are n yes votes (hence $n + 1$ no votes). Clearly, all proposals are adopted, since $3n/4n + 1 \geqslant \epsilon$, but a majority of voters (the first $2n + 1$ voters) disagree with the outcomes in a majority ($n + 1$ out of $2n + 1$) of cases.[3]

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NOTES

[1] In using the term 'coalition', we do not mean to suggest that members of a coalition engage in any sort of strategic cooperation. On our usage, the prevailing coalition on a given proposal is simply the set of voters who agree with the outcome of voting on that proposal, as determined by whatever decision procedure is employed. Prevailing coalitions do not necessarily comprise a majority of those voting. If, for example, adoption of a proposal requires the assent of more than two-thirds of those voting, a proposal may be rejected by a prevailing coalition of just one-third of those voting.

[2] When $N \leqslant 3$ or $K \leqslant 2$, and outcomes are determined by simple majorities, it is impossible to have a majority of individuals disagreeing with outcomes in a majority of cases. For an A–D matrix arising from application of majority rule always contains at least $[(N+1)/2]K$ A's and when $N \leqslant 3$ or $K \leqslant 2$, $[(N+1)/2]K > ([N/2]+1)[(K-1)/2] + [(N-1)/2]K$.

[3] These examples show also that the rule of three-fourths, applied in the case of simple majority rule, is the best possible single result covering all values of N and K. The example provided in Section 2 to show this was, however, of independent interest.

REFERENCES

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