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Generalized Stirling and Lah numbers

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To Leonard Carlitz in his eighty-ninth year

Abstract

The theory of modular binomial lattices enables the simultaneous combinatorial analysis of finite sets, vector spaces, and chains. Within this theory three generalizations of Stirling numbers of the second kind, and of Lah numbers, are developed.

1. Stirling numbers and their formal generalizations

The notational conventions of this paper are as follows: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{P} = \{1, 2, \dots\}$, $[0] = \emptyset$, and $[n] = \{1, \dots, n\}$ for $n \in \mathbb{P}$. Empty sums take the value 0 and empty products the value 1. Also, $x^0 = x^{\underline{0}} = x^{\overline{0}} = 1$ for all x (including $x = 0$), and for $n \in \mathbb{P}$, $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ and $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$.

As enumerator of partitions of $[n]$ with k blocks, the Stirling number of the second kind $S(n,k)$ plays a central role in elementary combinatorics. Not surprisingly, apart from the boundary values $S(n,0) = \delta_{n,0}$ and $S(n,k) = 0$ for $0 \leqslant n < k$, there are many representations of these numbers. From the standpoint of generalizations pursued in this paper these representations fall naturally into three classes:

Class I

$$S(n,k) = \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \in \mathbb{P}}} \frac{n!}{n_1! \cdots n_k!}, \tag{1.1}$$

$$S(n,k) = \frac{1}{k} \sum_{j=1}^{n} \binom{n}{j} S(n-j, k-1), \tag{1.2}$$

$$\sum_{n \geqslant 0} S(n,k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}, \tag{1.3}$$

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and, with $B_n := \sum_{k=0}^{n} S(n,k)$,

$$\sum_{n \geqslant 0} B_n \frac{x^n}{n!} = \mathrm{e}^{\mathrm{e}^x - 1}. \tag{1.4}$$

Class II

$$S(n,k) = \sum_{\substack{d_1 + \cdots + d_k = n-k \\ d_i \in N}} 1^{d_1} 2^{d_2} \cdots k^{d_k}, \tag{1.5}$$

$$\sum_{n \geqslant 0} S(n,k) x^n = x^k / \prod_{1 \leqslant j \leqslant k} (1 - jx), \tag{1.6}$$

$$S(n,k) = S(n-1, k-1) + kS(n-1, k), \tag{1.7}$$

$$x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}, \tag{1.8}$$

$$S(n,k) = \frac{\Delta^k 0^n}{k!} = \frac{\Delta^k x^n|_{x=0}}{k!}. \tag{1.9}$$

Class III

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \tbinom{k}{j} j^n, \tag{1.10}$$

$$S(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} S(j, k-1), \tag{1.11}$$

and, with B_n as above,

$$B_n = \frac{1}{\mathrm{e}} \sum_{k \geqslant 0} \frac{k^n}{k!}. \tag{1.12}$$

This paper develops three generalizations of $S(n,k)$ within the theory of modular binomial lattices, an important class of structures first identified by Doubilet et al. [9] as the ideal setting for the simultaneous combinatorial analysis of finite sets, vector spaces and chains. These generalizations encompass in particular the Bender–Goldman [2] and Carlitz–Milne [11] analogues of $S(n,k)$ for finite vector spaces. The generalizations of $S(n,k)$ in Sections 4 and 5 (and of the Lah numbers in Section 6) are combinatorial. In two instances, however, they are themselves special cases of the formal generalizations of $S(n,k)$ described below, a fact which greatly facilitates their analysis.

Theorem 1.1. *Given any sequence $(f_n)_{n\geqslant 0}$ of nonzero complex numbers with $f_0 = 1$, the following are equivalent characterizations of an array $(\mathscr{F}(n,k))_{n,k\geqslant 0}$:*

$$\mathscr{F}(n,k) = \frac{1}{k!}\sum_{\substack{n_1+\cdots+n_k=n\\ n_i\in\mathbb{P}}}\frac{f_n}{f_{n_1}\cdots f_{n_k}}, \quad \textit{for all } n\in\mathbb{N} \textit{ and } k\in\mathbb{P}, \tag{1.13}$$

with $\mathscr{F}(n,0) = \delta_{n,0}$ for all $n\in\mathbb{N}$,

$$\mathscr{F}(n,k) = \frac{1}{k}\sum_{j=1}^{n}\frac{f_n}{f_j f_{n-j}}\mathscr{F}(n-j,k-1) \quad \textit{for all } n,k\in\mathbb{P}, \tag{1.14}$$

with $\mathscr{F}(n,0) = \delta_{n,0}$ and $\mathscr{F}(0,k) = \delta_{0,k}$ for all n, $k\in\mathbb{N}$, and

$$\sum_{n\geqslant 0}\mathscr{F}(n,k)\frac{x^n}{f_n} = \frac{1}{k!}\left(\sum_{n\geqslant 1}\frac{x^n}{f_n}\right)^k, \quad \textit{for all } k\in\mathbb{N}. \tag{1.15}$$

Proof. The proof is a straightforward algebraic exercise. □

Judging from Ward's formal generalization of Bernoulli numbers [17], in which he mentions a similar generalization of Stirling numbers, it is likely that he had in mind numbers of the type $\mathscr{F}(n,k)$. Thus it seems appropriate to call such numbers *the Ward numbers associated with* $(f_n)_{n\geqslant 0}$. If $f_n = n!$, the $\mathscr{F}(n,k) = S(n,k)$ and (1.13)–(1.15) reduce to (1.1)–(1.3). Another example of Ward numbers, *the Bender–Goldman Stirling numbers of a modular binomial lattice*, will be developed in Section 4.

Theorem 1.2. *Given any sequence $(u_n)_{n\geqslant 0}$ of complex numbers, the following are equivalent characterizations of an array $(\mathscr{U}(n,k))_{n,k\geqslant 0}$:*

$$\mathscr{U}(n,k) = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k\\ d_i\in\mathbb{N}}} u_0^{d_0}u_1^{d_1}\cdots u_k^{d_k} \quad \textit{for all } n,k\in\mathbb{N}, \tag{1.16}$$

$$\sum_{n\geqslant 0}\mathscr{U}(n,k)x^n = \frac{x^k}{(1-u_0x)(1-u_1x)\cdots(1-u_kx)}, \quad \textit{for all } k\in\mathbb{N}, \tag{1.17}$$

$$\mathscr{U}(n,k) = \mathscr{U}(n-1,k-1) + u_k\mathscr{U}(n-1,k), \quad \textit{for all } n,k\in\mathbb{P}, \tag{1.18}$$

with $\mathscr{U}(n,0) = u_0^n$ *and* $\mathscr{U}(0,k) = \delta_{0,k}$ *for all $n,k\in\mathbb{N}$, and*

$$x^n = \sum_{k=0}^{n}\mathscr{U}(n,k)p_k(x), \quad \textit{for all } n\in\mathbb{N}, \tag{1.19}$$

with $p_0(x) := 1$ and $p_k(x) := (x-u_0)\cdots(x-u_{k-1})$, for all $k\in\mathbb{P}$.

Proof. The proof is a straightforward algebraic exercise. □

Since Comtet [6] observed that (1.19) implies (1.16)–(1.18), it seems appropriate to call the numbers $\mathscr{U}(n,k)$ *the Comtet numbers associated with* $(u_n)_{n\geqslant 0}$. If $u_n = n$, then

$\mathscr{U}(n,k) = S(n,k)$ and (1.16)–(1.19) reduce to (1.5)–(1.8). Another example of Comtet numbers, one of two varieties of the *Carlitz–Milne Stirling numbers of a modular binomial lattice*, will be developed in Section 5. The class of Comtet numbers encompasses not just Stirling numbers, but also binomial and q-binomial coefficients. For if $u_n \equiv 1$, then $\mathscr{U}(n,k) = \binom{n}{k}$, and if $u_n = q^n$, then $\mathscr{U}(n,k) = \binom{n}{k}_q$, as one easily sees from the recurrence (1.18) in these cases. In the latter case it follows from (1.16) that

$$\binom{n}{k}_q = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} = \sum_{t \geqslant 0} p(k, n-k, t) q^t, \tag{1.20}$$

where $p(k, n-k, t)$ denotes the number of partitions of the integer t with at most $n-k$ parts, each no larger than k.

2. Vector space analogues of Stirling numbers

The first q-Stirling numbers originated in Carlitz's beautiful paper [5] on q-Bernoulli numbers. Carlitz defined the former numbers, now denoted by $\tilde{S}_q(n,k)$, by means of the relations

$$(x_q)^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \tilde{S}_q(n,k)\, x_q (x-1)_q \cdots (x-k+1)_q, \tag{2.1}$$

with $x_q := (q^x - 1)/(q-1)$. He established the recurrence

$$\tilde{S}_q(n,k) = \tilde{S}_q(n-1,k-1) + k_q \tilde{S}_q(n-1,k), \tag{2.2}$$

as well as the explicit formula

$$\tilde{S}_q(n,k) = \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [(k-j)_q]^n / q^{\binom{k}{2}} k_q (k-1)_q \cdots 1_q, \tag{2.3}$$

employing in the proof of (2.3) a sequence of q-difference operators $\Delta_{q,k}$ defined recursively by

$$\Delta_{q,1} f(x) = \Delta f(x) = f(x+1) - f(x),$$
$$\Delta_{q,k+1} f(x) = \Delta_{q,k} f(x+1) - q^k \Delta_{q,k} f(x). \tag{2.4}$$

Carlitz construed q here as "an arbitrary parameter", noting that (1.8) is the limiting case of (2.1) as $q \to 1$ and, similarly, that (2.2) and (2.3) become, respectively, (1.7) and (1.10) when $q = 1$. In an earlier paper [4], however, he had proved that for odd primes p the quantity $(p-1)^{n-k}\, \tilde{S}_p(n,k)$ enumerates a certain class of abelian fields. So the origins of Carlitz's q-Stirling numbers are indirectly combinatorial.

Since the q-binomial coefficient $\binom{n}{k}_q$ counts the k-dimensional subspaces of the n-dimensional $GF(q)$-vector space $V_{q,n}$, the appearance of these coefficients in (2.3) suggests the possibility of a vector space interpretation of $\tilde{S}_q(n,k)$. The natural vector space analogues of set partitions – unordered direct sum decompositions of $V_{q,n}$ with k summands – are, however, *not* enumerated by $\tilde{S}_q(n,k)$. As Bender and Goldman [2] showed, the number, $\hat{S}_q(n,k)$, of such decompositions satisfies

$$\sum_{n \geqslant k} \hat{S}_q(n,k) \frac{x^n}{(q^n-1)\cdots(q^n-q^{n-1})} = \frac{1}{k!}\left\{\sum_{n \geqslant 1} \frac{x^n}{(q^n-1)\cdots(q^n-q^{n-1})}\right\}^k. \tag{2.5}$$

In particular, $\hat{S}_q(n,n) = q^{\binom{n}{2}} n_q(n-1)_q \cdots 1_q/n!$, whereas $\tilde{S}_q(n,n) = 1$.

Vector space interpretation of $\tilde{S}_q(n,k)$, and $S_q(n,k) = q^{\binom{k}{2}}\tilde{S}_q(n,k)$, were ultimately discovered by Milne [11], who began by formulating an inspired new interpretation of the classical Stirling numbers. Milne represented each k block partition of $[n]$ by a canonical ordered partition $(E_1,\ldots,E_k)$ of $[n]$, with $\min\{i \in E_j\} < \min\{i \in E_{j+1}\}$, $j = 1,\ldots,k-1$, and associated with each such ordered partition a function $f:[n] \to [k]$, with $f(i) = j$ for all $i \in E_j$. The resulting *restricted growth functions from* $[n]$ *to* $[k]$, i.e., surjections $f:[n] \to [k]$ such that in a left-to-right scan of $(f(1),\ldots,f(n))$ the first occurrence of j precedes the first occurrence of $j+1$, for $j = 1,\ldots,k-1$, are thus also enumerated by $S(n,k)$. Two q-analogues of restricted growth functions, each involving certain sequences $(U_1,\ldots,U_n)$ of one-dimensional subspaces of $V_{q,k}$, then turn out, *mirabile dictu*, to be enumerated by precisely $\tilde{S}_q(n,k)$ and $S_q(n,k)$.

In the context of modular binomial lattices a common generalization of $S(n,k)$ and $\hat{S}(n,k)$ will be developed in Section 4, and common generalizations of $S(n,k)$ and $\tilde{S}_q(n,k)$, and of $S(n,k)$ and $S_q(n,k)$ in Section 5. Analogous generalizations of the Lah numbers will be developed in Section 6. The next section offers a brief review of the theory of modular binomial lattices.

3. Modular binomial lattices (q-lattices)

The theory of binomial posets is treated in [14, chapter 3], and various aspects of modular binomial lattices in [8,9,16]. The following is a brief review of pertinent results.

A poset P is called a *binomial poset* if it satisfies the following three conditions:

(I) P is locally finite with $\hat{0}$ and contains an infinite chain.

(II) Every interval $[x,y]$ of P is *graded*, i.e., all maximal chains in $[x,y]$ have the same length. If this common length is n, write $l(x,y) = n$ and call $[x,y]$ and an *n-interval*.

(III) For all $n \in \mathbb{N}$, any two n-intervals contain the same number, $B(n)$, of maximal chains.

As a consequence of (II), each n-interval $[x,y]$ of a binomial poset admits a rank function $\rho_{x,y}:[x,y] \to \{0,1,\ldots,n\}$, written simply as ρ if no confusion results, defined by $\rho(z) = l(x,z)$. If $x \leqslant u \leqslant v \leqslant y$, then u is *covered by* v, i.e., $|[u,v]| = 2$, iff $\rho(v) = \rho(u) + 1$. The only rank 0 element of $[x,y]$ is x, and the only rank n element is y. The *atoms* of $[x,y]$ are all its rank 1 elements, i.e., all elements that cover x.

By (II) and (III) any two n-intervals in a binomial poset contain the same number, $\left[{n \atop k}\right]$, of elements of rank k, where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{B(n)}{B(k)B(n-k)}, \tag{3.1}$$

whence

$$B(n) = \begin{bmatrix} n \\ 1 \end{bmatrix} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \cdots \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.2}$$

The quantities $\left[{n \atop k}\right]$ are called the *incidence coefficients* of P.

A *binomial lattice* is simply a lattice that is a binomial poset. An important class of such lattices, the *modular binomial lattices* are those for which the rank function ρ of each interval $[x, y]$ satisfies

$$\rho(w \vee z) = \rho(w) + \rho(z) - \rho(w \wedge z), \quad \text{for all } w, z \in [x, y]. \tag{3.3}$$

Doubilet et al. [9] have shown that if L is a modular binomial lattice and one defines the *characteristic* q of L by

$$q := \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1, \tag{3.4}$$

then the number of atoms $\left[{n \atop 1}\right]$ in any n-interval of L is given by

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = n_q := 1 + q + \cdots + q^{n-1}, \quad \text{with } 0_q := 0, \tag{3.5}$$

and so, by (3.2), the number of maximal chains $B(n)$ in any n-interval of L is given by

$$B(n) = n_q^! := n_q(n-1)_q \cdots 1_q, \quad \text{with } 0_q^! := 1. \tag{3.6}$$

By (3.1) and (3.6) the incidence coefficient $\left[{n \atop k}\right]$ of a modular binomial lattice of characteristic q is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}_q := \frac{n_q^!}{k_q^!(n-k)_q^!}, \quad 0 \leqslant k \leqslant n. \tag{3.7}$$

The theory of modular binomial lattices provides the setting for a simultaneous combinatorial analysis of

(1) the lattice of finite subsets of an infinite set, with $\rho_{x,y}(z) = |z \backslash x|$ and $q = 1$,

(2) the lattice of finite subspaces of an infinite vector space over the finite field of order p^d, with $\rho_{x,y}(z) = dim(z/x)$ and $q = p^d$,

(3) the chain $(\mathbb{N}, \leqslant)$, with $\rho_{x,y}(z) = z - x$ and $q = 0$.

Indeed, by [9, Theorem 8.2] and [3, IV.13, par. 4], these are the only modular binomial lattices. Henceforth, a modular binomial lattice of characteristic q will simply be called a q-lattice. The following theorem introduces a q-lattice generalization of $n!$ that is an important variant of $n_q^!$.

Theorem 3.1. *Let* $[x, y]$ *be any n-interval of a q-lattice, and let* $\mathscr{A} = \{(\alpha_1, \ldots, \alpha_n)$: *each* α_i *is an atom of* $[x, y]$ *and* $\alpha_1 \vee \cdots \vee \alpha_n = y\}$. *Then*

$$|\mathscr{A}| = n_q^{\underline{n}} := q^{\binom{n}{2}} n_q^!, \text{ with } 0_q^{\underline{0}} := 1. \tag{3.9}$$

Proof. Let $\mathscr{B} = \{(\beta_0, \beta_1, \ldots, \beta_n): x = \beta_0 < \beta_1 < \cdots < \beta_n = y\}$, the set of maximal chains in $[x, y]$. Equivalently, $(\beta_0, \ldots, \beta_n) \in \mathscr{B}$ iff $\beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_n$ and $\rho(\beta_i) = i$, $0 \leqslant i \leqslant n$, where ρ is the rank function of $[x, y]$. The modular identity (3.3) implies that for every $\beta \in [x, y]$ and every atom $\alpha \in [x, y]$, $\rho(\beta \vee \alpha) \leqslant \rho(\beta) + 1$, with equality iff $\alpha \nleqslant \beta$ (although only semi-modularity appears to be used here, semi-modularity implies modularity in binomial lattices [9, Lemma 8.2]).

Thus if $(\alpha_1, \ldots, \alpha_n) \in \mathscr{A}$, then $\rho(\alpha_1 \vee \cdots \vee \alpha_i) = i$ for all $i \in [n]$, since $\rho(\alpha_1 \vee \cdots \vee \alpha_n) = \rho(y) = n$ and joining an atom to an element of $[x, y]$ raises its rank by at most 1. So $(\alpha_1, \ldots, \alpha_n) \mapsto (x, \alpha_1, \alpha_1 \vee \alpha_2, \ldots, \alpha_1 \vee \cdots \vee \alpha_n)$ is a map from $\mathscr{A}$ into $\mathscr{B}$.

Now the preimages of a given $(\beta_0, \ldots, \beta_n) \in \mathscr{B}$ under this map are precisely those sequences $(\alpha_1, \ldots, \alpha_n)$ of atoms of $[x, y]$ such that, for each $i \in [n]$, α_i is an element of the i-interval $[x, \beta_i]$, but not of the (i − 1)-interval $[x, \beta_{i-1}]$. By (3.5) there are $i_q - (i-1)_q = q^{i-1}$ such α_i, and thus $\prod_{1 \leqslant i \leqslant n} q^{i-1} = q^{\binom{n}{2}}$ such preimages. Hence by (3.6), $|\mathscr{A}| = q^{\binom{n}{2}} |\mathscr{B}| = q^{\binom{n}{2}} n_q^! = n_q^{\underline{n}}$. □

Note that one may also write $n_q^{\underline{n}} = n_q(n_q - 1_q) \cdots (n_q - (n-1)_q)$, which explains the notation chosen for this quantity.

4. The Bender–Goldman Stirling numbers of a q-lattice

The following theorem provides the foundation for a common generalization of set partitions and direct sum decompositions of a vector space.

Theorem 4.1. *Let* $[x, y]$ *be an interval in a q-lattice, with rank function* ρ, *and let* $(z_1, \ldots, z_k)$ *be a sequence in the half-open interval* $(x, y]$ *such that*

$$z_1 \vee \cdots \vee z_k = y. \tag{4.1}$$

The following conditions are then equivalent:

$$\left(\bigvee_{i \neq j} z_i \right) \wedge z_j = x \quad \text{for all } j \in [k], \tag{4.2}$$

$$(z_1 \vee \cdots \vee z_j) \wedge z_{j+1} = x \quad \text{for all } j \in [k-1], \tag{4.3}$$

$$\rho(z_1) + \cdots + \rho(z_k) = \rho(y), \tag{4.4}$$

$$\rho\left(\bigvee_{i \in I} z_i \right) = \sum_{i \in I} \rho(z_i) \quad \text{for all nonempty } I \subset [k]. \tag{4.5}$$

Proof. Obviously (4.2) implies (4.3), and (4.5) implies (4.4). Repeated application of (3.3) yields

$$\rho(z_1 \vee \cdots \vee z_k) = \sum_{j=1}^{k} \rho(z_j) - \sum_{j=1}^{k-1} \rho((z_1 \vee \cdots \vee z_j) \wedge z_{j+1}), \tag{4.6}$$

and so by (4.1), it is clear that (4.3) implies (4.4).

By straightforward induction on $|I|$, (4.2) implies (4.5). In showing, finally, that (4.4) implies (4.2), there is no loss in generality in assuming that $j = k$, for relabeling the z_i does not change (4.4). Combining (4.4) with (4.6) yields in particular that $\rho((z_1 \vee \cdots \vee z_{k-1}) \wedge z_k) = 0$, and so $(z_1 \vee \cdots \vee z_{k-1}) \wedge z_k = x$, as desired. □

A sequence $(z_1, \ldots, z_k)$ in $(x, y]$ satisfying (4.1) and any (hence all) of the conditions (4.2)–(4.5) will be called an *ordered direct k-sum decomposition of y in* $(x, y]$. A set $S \subset (x, y]$ such that $|S| = k$, $\vee S = y$, and $(\vee S \backslash z) \wedge z = x$ for all $z \in S$ will be called a *direct k-sum decomposition of y in* $(x, y]$. The number of such decompositions depends only on the length of $[x, y]$, which will first be proved for modular binomial lattices of positive characteristic.

Theorem 4.2. *If* $[x, y]$ *is an n-interval in a q-lattice, where* $q > 0$, *then the number of direct k-sum decomposition of y in* $(x, y]$ *is given by* $\hat{S}_q(n, k)$, *where*

$$\hat{S}_q(n, k) = \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \in \mathbb{P}}} \frac{n_q^{\underline{n}}}{n_1{}_q^{\underline{n_1}} \cdots n_k{}_q^{\underline{n_k}}} \quad \text{for all } n \in \mathbb{N} \text{ and } k \in \mathbb{P}, \tag{4.7}$$

with $\hat{S}_q(n, 0) = \delta_{n,0}$ *for all* $n \in \mathbb{N}$,

$$\hat{S}_q(n, k) = \frac{1}{k} \sum_{j=1}^{n} \frac{n_j^{\underline{n}}}{j_q^{\underline{j}} (n - j)_q^{\underline{(n-j)}}} \hat{S}_q(n - j, k - 1) \quad \text{for all } n, k \in \mathbb{P}, \tag{4.8}$$

with $\hat{S}_q(n, 0) = \delta_{n,0}$ *and* $\hat{S}_q(0, k) = \delta_{0,k}$, *for all* $n, k \in \mathbb{N}$, *and*

$$\sum_{n \geqslant 0} \hat{S}_q(n, k) \frac{x^n}{n_q^{\underline{n}}} = \frac{1}{k!} \left(\sum_{n \geqslant 1} \frac{x^n}{n_q^{\underline{n}}} \right)^k \quad \text{for all } k \in \mathbb{N}. \tag{4.9}$$

Proof. Since the supremum of the empty subset of $[x, y]$ is x, the number of direct 0-sum decompositions is clearly $\delta_{n,0}$.

If $k \in \mathbb{P}$, the summands z_i of an *ordered* decomposition are distinct by (4.2). This fact, along with (4.4), reduces the proof of (4.7) to showing that for positive integers $n_1 + \cdots + n_k = n$, if $\mathscr{D}$ is the family of all ordered direct k-sum decompositions $(z_1, \ldots, z_n)$ of y in $(x, y]$, with $\rho(z_i) = n_i$, then

$$|\mathscr{D}| = \frac{n_q^{\underline{n}}}{n_1{}_q^{\underline{n_1}} \cdots n_k{}_q^{\underline{n_k}}}. \tag{4.10}$$

Let $\mathscr{A} = \{(\alpha_1, \dots, \alpha_n)$: each α_i is an atom of $[x,y]$ and $\alpha_1 \vee \cdots \vee \alpha_n = y\}$. Each member of $\mathscr{A}$ satisfies (4.1) and (4.4), and thus (4.5), for $k = n$. So $(\alpha_1, \dots, \alpha_n) \mapsto (z_1, \dots, z_k)$, with $z_1 = \alpha_1 \vee \cdots \vee \alpha_{n_1}$, $z_2 = \alpha_{n_1+1} \vee \cdots \vee \alpha_{n_2}$, etc., is a map from $\mathscr{A}$ into $\mathscr{D}$. By Theorem 3.1, $|\mathscr{A}| = n_q^{\underline{n}}$ and, moreover, each element of $\mathscr{D}$ has $n_1{}_q^{\underline{n_1}} \cdots n_k{}_q^{\underline{n_k}}$ preimages in $\mathscr{A}$. This establishes (4.10), and thus (4.7).

But now it is clear that for $q > 0$, $\hat{S}_q(n,k)$ is a Ward number associated with $f_n = n_q^{\underline{n}}$. So (4.8) and (4.9) follow immediately from (4.7) and Theorem 1.1. □

If one defines $\hat{B}_{q,n}$ by

$$\hat{B}_{q,n} := \sum_{k=0}^{n} \hat{S}_q(n,k), \tag{4.11}$$

then (4.9) implies that

$$\sum_{n \geqslant 0} \hat{B}_{q,n} \frac{x^n}{n_q^{\underline{n}}} = \exp\left(\sum_{n \geqslant 1} \frac{x^n}{n_q^{\underline{n}}}\right), \tag{4.12}$$

which reduces to (1.4) when $q = 1$. Thus for $q > 0$, $\hat{S}_q(n,k)$ is a generalization of $S(n,k)$ for which generalizations of all the Class I properties of the latter exist.

Since $n_q^{\underline{n}} = q^{\binom{n}{2}} n_q^!$, (4.7) may be rewritten as

$$\hat{S}_q(n,k) = \frac{1}{k!} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_i \in \mathbb{P}}} q^{\sum_{j=1}^{k-1} n_j(n - n_1 - \cdots - n_j)} \frac{n_q^!}{n_{1q}^! \cdots n_{k_q}^!} \tag{4.13}$$

and (4.8) as

$$\hat{S}_q(n,k) = \frac{1}{k} \sum_{j=1}^{n} q^{j(n-j)} \binom{n}{j}_q \hat{S}_q(n-j, k-1). \tag{4.14}$$

The advantage of such a rewriting is that (4.13) and (4.14) hold for all q, including $q = 0$. For it is easy to check that if $[x,y]$ is an n-interval in the 0-lattice $\mathbb{N}$, then the number of direct k-sum decompositions of y in $(x,y]$ is equal to one when $n = k = 0$ and when $n \geqslant k = 1$, and zero otherwise.

It is perhaps worth noting that (4.13) and (4.14) can each be established by arguments that hold for all q, rather than by separate arguments for $q > 0$ and $q = 0$. One uses in each case the fact [8, Theorem 4.1] that in any q-lattice, if z is an element of rank j in an n-interval $[x,y]$, then z has $q^{j(n-j)}$ *complements* in $[x,y]$, i.e., elements w such that $z \wedge w = x$ and $z \vee w = y$.

5. The Carlitz–Milne Stirling numbers of a q-lattice

Let $[x,y]$ be a k-interval in an arbitrary q-lattice, with $n \in \mathbb{N}$, and define the families $\mathscr{S}, \mathscr{A}$, and $\mathscr{B}$ as follows:

$$\mathscr{S} = \{(a_1, \dots, a_n)\text{: each } a_i \text{ is an atom of } [x,y] \text{ and } a_1 \vee \cdots \vee a_n = y\}, \tag{5.1}$$

$$\mathscr{A} = \{(\alpha_1, \dots, \alpha_k)\text{: each } \alpha_j \text{ is an atom of } [x,y] \text{ and } \alpha_1 \vee \cdots \vee \alpha_k = y\}, \tag{5.3}$$

$$\mathscr{B} = \{(\beta_0, \dots, \beta_k)\text{: } x = \beta_0 < \beta_1 < \cdots < \beta_k = y\}. \tag{5.3}$$

By (3.6), $|\mathscr{B}| = k_q^!$ and by Theorem 3.1, $|\mathscr{A}| = q^{\binom{k}{2}} k_q^! = k_q^{\underline{k}}$. Based on the theory of covering algebras, it was shown in [16] that, in any binomial lattice, $|\mathscr{S}|$ depends only on n and k, indeed that

$$|\mathscr{S}| = \sum_{j=0}^{k} M(j) \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} k-j \\ 1 \end{bmatrix}^n, \tag{5.4}$$

where $M(j)$ is the value of the Möbius function of that lattice on any j-interval. In particular, if the lattice in question is a q-lattice, one has

$$|\mathscr{S}| = \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [(k-j)_q]^n. \tag{5.5}$$

While parts of the following analysis could be based on (5.5) and other results from [16], we have opted instead for a more elementary, self-contained treatment. The fact that $|\mathscr{S}|$ depends only on n and k in the case of q-lattices will be a corollary of this analysis. The q-Stirling numbers $\tilde{S}_q(n,k)$ and $S_q(n,k)$ will arise from certain natural mappings from $\mathscr{S}$ to $\mathscr{A}$ and from $\mathscr{S}$ to $\mathscr{B}$.

Given a k-interval $[x,y]$ of a q-lattice, with rank function ρ, the modular identity (3.3) implies that for every $(a_1,\dots,a_n) \in \mathscr{S}$, $\{\rho(a_1 \vee \cdots \vee a_i)\colon i \in [n]\} = [k]$. (For if not, let $l = \max [k] \setminus \{\rho(a_1 \vee \cdots \vee a_i)\colon i \in [n]\}$ and $t = \min\{i\colon \rho(a_1 \vee \cdots \vee a_i) = l+1\}$. Then $\rho(a_1 \vee \cdots \vee a_{t-1}) \leqslant l-1$ and so $\rho(a_1 \vee \cdots \vee a_t) \leqslant l$, a contradiction.) In particular, this statement is vacuously true if $n < k$, for since $\rho(a_1 \vee \cdots \vee a_n) \leqslant n$, $\mathscr{S} = \emptyset$ in this case.

In light of the above observations, the function

$$(a_1,\dots,a_n) \mapsto (a_{i_1}, a_{i_2}, \dots, a_{i_k}), \tag{5.6}$$

with $i_j := \min\{i \in [n]\colon \rho(a_1 \vee \cdots \vee a_i) = j\}, j \in [k]$, is well defined for all $(a_1,\dots,a_n) \in \mathscr{S}$.

Theorem 5.1. *The function* $(a_1,\dots,a_n) \mapsto (a_{i_1},\dots,a_{i_k})$ *defined by* (5.6) *maps* $\mathscr{S}$ *into* $\mathscr{A}$ *and every element of* $\mathscr{A}$ *has* $\tilde{S}_q(n,k)$ *preimages in* $\mathscr{S}$ *with respect to this function, where*

$$\tilde{S}_q(n,k) := \sum_{\substack{d_1 + \cdots + d_k = n-k \\ d_i \in \mathbb{N}}} (1_q)^{d_1} (2_q)^{d_2} \cdots (k_q)^{d_k}. \tag{5.7}$$

Proof. Since y is the only rank k element of $[x,y]$, to show that (5.6) defines a map into $\mathscr{A}$, it suffices to show that for every $(a_1,\dots,a_n) \in \mathscr{S}$, $a_{i_1} \vee \cdots \vee a_{i_k} = a_1 \vee a_2 \vee \cdots \vee a_{i_k}$.

One proves this by showing by induction on j that

$$a_{i_1} \vee \cdots \vee a_{i_j} = a_1 \vee a_2 \vee \cdots \vee a_{i_j} \quad \text{for all } j \in [k]. \tag{5.8}$$

Since $i_1 = 1$, (5.8) holds for $j = 1$. Given (5.8) for some $j \in [k-1]$, one has, by modularity and the definition of i_{j+1} that $a_1 \vee a_2 \vee \cdots \vee a_{i_j} = a_1 \vee a_2 \vee \cdots \vee a_{i_{j+1}-1}$. Hence, $a_{i_1} \vee \cdots \vee a_{i_{j+1}} = a_1 \vee a_2 \vee \cdots \vee a_{i_{j+1}}$.

Now all of the preimages $(a_1, \ldots, a_n)$ of a given $(\alpha_1, \ldots, \alpha_k) \in \mathscr{A}$ under the map (5.6) may be constructed by

(1) choosing a sequence $(i_1, \ldots, i_k)$ with $1 = i_1 < i_2 < \cdots < i_k \leqslant n$, and setting $a_{i_j} = \alpha_j$ for all $j \in [k]$, and

(2) for all $j \in [k-1]$, choosing for the $d_j := i_{j+1} - i_j - 1$ elements a_i, where $i_j < i < i_{j+1}$, arbitrary atoms of the j-interval $[x, \alpha_1 \vee \cdots \vee \alpha_j]$, and for the $d_k := n - i_k$ elements a_i, where $i_k < i \leqslant n$, arbitrary atoms of the k-interval $[x, \alpha_1 \vee \cdots \vee \alpha_k] = [x, y]$.

By (3.5), this construction may be effected in $\tilde{S}_q(n,k)$ ways. □

By (5.7), $\tilde{S}_q(n,k)$ is a Comtet number with $u_n = n_q$. Hence by Theorem 1.2,

$$\sum_{n \geqslant 0} \tilde{S}_q(n,k) x^n = \frac{x^k}{(1 - 0_q x)(1 - 1_q x) \cdots (1 - k_q x)}, \quad \text{for all } k \in \mathbb{N}, \tag{5.9}$$

$$\tilde{S}_q(n,k) = \tilde{S}_q(n-1, k-1) + k_q \tilde{S}_q(n-1, k), \quad \text{for all } n, k \in \mathbb{P}, \tag{5.10}$$

with $\tilde{S}_q(n,0) = \delta_{n,0}$ and $\tilde{S}_q(0,k) = \delta_{0,k}$ for all $n, k \in \mathbb{N}$, and

$$x^n = \sum_{k=0}^{n} \tilde{S}_q(n,k) \varphi_k(x), \quad \text{for all } n \in \mathbb{N}, \tag{5.11}$$

with $\varphi_0(x) := 1$ and $\varphi_k(x) := x(x - 1_q) \cdots (x - (k-1)_q)$ for all $k \in \mathbb{P}$.

If, following Davis [7], one defines the q-difference $\Delta_q : \mathbb{C}[x] \to \mathbb{C}[x]$ by

$$\Delta_q p(x) = \frac{p(qx+1) - p(x)}{(q-1)x + 1}, \tag{5.12}$$

then

$$\Delta_q \varphi_k(x) = k_q \varphi_{k-1}(x) \quad \text{for all } k \in \mathbb{N}. \tag{5.13}$$

From (5.11) and (5.13) it follows that

$$\tilde{S}_q(n,k) = \frac{\Delta_q^k x^n |_{x=0}}{k_q^!}, \tag{5.14}$$

where Δ_q^k is the k-fold composition of Δ_q (cf. (2.4)). So $\tilde{S}_q(n,k)$ is a generalization of $S(n,k)$ for which generalizations of all of the Class II properties of the latter exist.

The case $q = 0$ of the above is somewhat intriguing. By (5.9) or (5.10) $\tilde{S}_0(n,0) = \delta_{n,0}$ and $\tilde{S}_0(0,k) = \delta_{0,k}$ for all $n, k, \in \mathbb{N}$, and $\tilde{S}_0(n,k) = \binom{n-1}{k-1}$ for all $n, k \in \mathbb{P}$. Of course, if $[x,y]$ is a k-interval in the 0-lattice $(\mathbb{N}, \leqslant)$, the set $\mathscr{S}$ defined by (5.1) is nonempty (and then of cardinality 1) iff $n = k = 0$ or $n \geqslant k \geqslant 1$. Similarly, the set $\mathscr{A}$ defined by (5.2) is nonempty (and of cardinality 1) iff $k = 0$ or $k = 1$. In particular, if $k \geqslant 2$ the map defined by (5.6) is the empty function from $\emptyset$ to $\emptyset$. In this case, the claim that each element of $\mathscr{A}$ has $\binom{n-1}{k-1}$ preimages is true, as claimed in Theorem 5.1, but vacuously so.

The q-Stirling numbers $S_q(n,k)$ arise in connection with the function

$$(a_1,\dots,a_n)\mapsto (x,(a_1\vee a_2\vee\cdots\vee a_{i_j})_{1\leqslant j\leqslant k}) \tag{5.15}$$

with $i_j := \min\{i\in[n]\colon \rho(a_1\vee\cdots\vee a_i)=j\}$.

Theorem 5.2. *The function defined by* (5.15) *maps* $\mathscr{S}$ *into* $\mathscr{B}$, *and every element of* $\mathscr{B}$ *has*

$$S_q(n,k) := q^{\binom{k}{2}}\tilde{S}_q(n,k) \tag{5.16}$$

preimages in $\mathscr{S}$ *with respect to this function.*

Proof. By (5.8), the function given by (5.15) is identical with that given by $(a_1,\dots,a_n)\mapsto(x,(a_{i_1}\vee a_{i_2}\vee\cdots\vee a_{i_j})_{1\leqslant j\leqslant k})$. Thus it is just the composition of the map from $\mathscr{S}$ to $\mathscr{A}$ given by (5.6) with the map from $\mathscr{A}$ to $\mathscr{B}$ given by $(\alpha_1,\dots,\alpha_k)\mapsto(x,(\alpha_1\vee\alpha_2\vee\cdots\vee\alpha_j)_{1\leqslant j\leqslant k})$. By the proof of Theorem 3.1 each element of $\mathscr{B}$ has $q^{\binom{k}{2}}$ preimages in $\mathscr{A}$ with respect to the latter map. Combined with Theorem 5.1, this yields (5.16). □

Of course, multiplying (5.9) and (5.10) by $q^{\binom{k}{2}}$ yields variants of those formulas for $S_q(n,k)$. In particular,

$$S_q(n,k) = q^{k-1}S_q(n-1,k-1) + k_qS_q(n-1,k) \quad \text{for all } n,k,\in\mathbb{P}. \tag{5.17}$$

Of more interest, however, are the following distinctive properties of $S_q(n,k)$.

Theorem 5.3. *For every q-lattice,*

$$S_q(n,k) = \frac{1}{k_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\binom{k}{j}_q (j_q)^n. \tag{5.18}$$

Proof. One can show [14, Theorem 3.6] by induction that if $q>0$ and $p(x)\in\mathbb{C}[x]$, then

$$\Delta_q^k p(x) = \frac{\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\binom{k}{j}_q p(q^jx+j_q)}{q^{\binom{k}{2}}[(q-1)x+1]^k}. \tag{5.19}$$

With (5.14) this implies that for $q>0$,

$$\tilde{S}_q(n,k) = \frac{\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\binom{k}{j}_q(q^j)^n}{q^{\binom{k}{2}}k_q!}. \tag{5.20}$$

and multiplying (5.20) by $q^{\binom{k}{2}}$ then yields (5.18) for $q>0$. But one may check that (5.18) also holds for $q=0$, for $S_0(n,k)$ is nonzero (and equal to 1) iff $n=k=0$ or $n\geqslant k=1$, which agrees with the right-hand side of (5.18). □

Theorem 5.4. *For every q-lattice,*

$$S_q(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} q^j S_q(j,k-1), \quad \text{for all } n \in \mathbb{N} \text{ and } k \in \mathbb{P}. \tag{5.21}$$

Proof. Using the values of $S_0(n,k)$ noted in the preceding paragraph, one may check that (5.21) holds for $q = 0$. The method of linear functionals [1, pp. 89–90], originated by Rota, will be used to treat the case $q > 0$.

By (5.11) and (5.16),

$$x^n = \sum_{j=0}^{n} S_q(n,j)\psi_j(x), \tag{5.22}$$

where $\psi_j(x) = \varphi_j(x)/q^{\binom{j}{2}}$. Note that

$$\psi_{j+1}(x) = x\psi_j\left(\frac{x-1}{q}\right). \tag{5.23}$$

For all $k \in \mathbb{N}$, define $L_k : \mathbb{C}[x] \to \mathbb{C}$ by setting $L_k\psi_j(x) = \delta_{k,j}$ and extending L_k to $\mathbb{C}[x]$ by linearity. By (5.22), $L_k(x^n) = S_q(n,k)$. By the definition of L_k and by (5.23), one has, for all $j \in \mathbb{N}$,

$$L_{k-1}(\psi_j(x)) = L_k(\psi_{j+1}(x)) = L_k\left(x\psi_j\left(\frac{x-1}{q}\right)\right), \tag{5.24}$$

and thus, for all $p(x) \in \mathbb{C}[x]$,

$$L_{k-1}(p(x)) = L_k\left(xp\left(\frac{x-1}{q}\right)\right). \tag{5.25}$$

Setting $p(x) = (qx+1)^n$ in (5.25) yields (5.21). □

From (5.21) it follows that the q-Bell numbers

$$B_{q,n} := \sum_{k=0}^{n} S_q(n,k) \tag{5.26}$$

satisfy the recurrence

$$B_{q,n+1} = \sum_{j=0}^{n} \binom{n}{j} q^j B_{q,j}, \tag{5.27}$$

which was proved by Milne [11] for the case $q = p^d$, based on (2.1). Our proof of (5.21) is very much in the spirit of Milne's proof, but being based on (5.22), has the advantage of holding for $q = 1$ as well as for $q = p^d$. The same remarks apply to the following Dobinski formula for $B_{q,n}$, established by Milne for the case $q = p^d$.

Theorem 5.5. *For every q-lattice with $q > 0$,*

$$B_{q,n} = \frac{1}{e_q(1)} \sum_{k \geqslant 0} \frac{(k_q)^n}{k_q^!}, \tag{5.28}$$

where

$$e_q(x) := \sum_{k \geqslant 0} \frac{x^k}{k_q^!}. \tag{5.29}$$

Proof. Define $L: \mathbb{C}[x] \to \mathbb{C}$ by $L(\psi_j(x)) = 1$ for all $j \in \mathbb{N}$, extending L to $\mathbb{C}[x]$ by linearity. By (5.22) and (5.26), $L(x^n) = B_{q,n}$.

For all $n \in \mathbb{N}$,

$$\begin{aligned} L(\psi_n(x)) = 1 &= \frac{1}{e_q(1)} \sum_{k \geqslant 0} \frac{1}{k_q^!} = \frac{1}{e_q(1)} \sum_{k \geqslant n} \frac{1}{(k-n)_q^!} \\ &= \frac{1}{e_q(1)} \sum_{k \geqslant n} \frac{\psi_n(k_q)}{k_q^!} = \frac{1}{e_q(1)} \sum_{k \geqslant 0} \frac{\psi_n(k_q)}{k_q^!}. \end{aligned} \tag{5.30}$$

Hence for all $p(x) \in \mathbb{C}[x]$,

$$L(p(x)) = \frac{1}{e_q(1)} \sum_{k \geqslant 0} \frac{p(k_q)}{k_q^!}. \tag{5.31}$$

Setting $p(x) = x^n$ in (5.31) yields (5.28). □

So $S_q(n,k)$ is a generalization of $S(n,k)$ for which generalizations of all of the Class III properties of the latter exist.

Milne [12] showed that $S_q(n,k)$, regarded as a polynomial in the indeterminate q, is the generating function for a simple statistic on $RGF(n,k)$, the set of restricted growth functions from $[n]$ to $[k]$. Specifically, for $f \in RGF(n,k)$, let

$$I^m(f) := \sum_{i \geqslant 1}^{n} I_i^m(f), \tag{5.32}$$

where $I_i^m(f)$ is the number of distinct integers among $f(1), \ldots, f(i-1)$ that are strictly less than $f(i)$. Milne proved that

$$\sum_{f \in RGF(n,k)} q^{I^m(f)} = S_q(n,k) \tag{5.33}$$

by showing that the left-hand side of (5.33) satisfies the recurrence (5.17).

Wachs and White [15], on the other hand, showed by a rook placement model that

$$I(f) := \sum_{i=1}^{n} f(i) = I^m(f) + n, \tag{5.34}$$

which, with (5.33), implies

$$\sum_{f \in RGF(n,k)} q^{I(f)} = q^n S_q(n,k). \tag{5.35}$$

It turns out that (5.33) and (5.35) are simple corollaries of a different sort of q-counting of $RGF(n,k)$, one which invokes the combinatorial interpretation of $\tilde{S}_q(n,k)$ established in this paper.

Theorem 5.6. *For $f \in RGF(n,k)$, let*

$$\tilde{I}(f) = \sum_{j=1}^{k} (j-1)(v_j(f)-1), \tag{5.36}$$

where

$$v_j(f) = |f^{-1}(\{j\})| \quad \text{for all } j \in [k]. \tag{5.37}$$

Then

$$\sum_{f \in RGF(n,k)} q^{\tilde{I}(f)} = \tilde{S}_q(n,k). \tag{5.38}$$

Proof. It suffices to prove (5.38) when q is the characteristic of an arbitrary modular binomial lattice L. Let $[x,y]$ be a k-interval in the q-lattice L. Fix $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathscr{A}$, as defined by (5.2), and let $\mathscr{S}_\alpha$ be the subset of $\mathscr{S}$, as defined by (5.1), consisting of all preimages of α under the map (5.6).

By Theorem 5.1, $|\mathscr{S}_\alpha| = \tilde{S}_q(n,k)$. We prove (5.38) by exhibiting a function from $\mathscr{S}_\alpha$ to $RGF(n,k)$ such that each $f \in RGF(n,k)$ has $q^{\tilde{I}(f)}$ preimages under this function. Let $(a_1, \ldots, a_n) \in \mathscr{S}_\alpha$, with $i_j := \min\{i\colon \rho(a_1 \vee \cdots \vee a_i = j\}$ for all $j \in [k]$. Consider the function

$$(a_1, \ldots, a_n) \mapsto (\tau(a_1), \ldots, \tau(a_n)), \tag{5.39}$$

where

$$\tau(a_i) = \min\{j\colon a_i \leqslant a_1 \vee \cdots \vee a_{i_j}\}. \tag{5.40}$$

Clearly, (5.39) maps $\mathscr{S}_\alpha$ into $RGF(n,k)$, with $\tau(a_{i_j}) = j$ for all $j \in [k]$. All of the preimages $(a_1, \ldots, a_n)$ of a given $f \in RGF(n,k)$ under the function (5.39) may be constructed by

(1) setting $a_i = \alpha_1$ for every i such that $f(i) = 1$,

(2) for $2 \leqslant j \leqslant k$, if i is the smallest element of $[n]$ with $f(i) = j$, setting $a_i = \alpha_j$; and if i is any other element of $[n]$ with $f(i) = j$, setting a_i equal to any of the $j_q - (j-1)_q = q^{j-1}$ atoms that belong to the j-interval $[x, \alpha_1 \vee \cdots \vee \alpha_j]$ but not to the $(j-1)$-interval $[x, \alpha_1 \vee \cdots \vee \alpha_{j-1}]$.

This construction may be effected in

$$\prod_{j=1}^{k} q^{(j-1)(v_j(f)-1)} = q^{\tilde{I}(f)} \tag{5.41}$$

ways, which proves (5.38). □

Clearly, $I^m(f) = \sum_{j=1}^{k} (j-1)v_j(f) = \tilde{I}(f) + \binom{k}{2}$ and $I(f) = \tilde{I}(f) + \binom{k}{2} + n$, and so (5.38), along with (5.16), implies both (5.33) and (5.35).

It should be noted that our conceptualizations of $\tilde{S}_q(n,k)$ and $S_q(n,k)$ are deeply indebted to Milne's conception of restricted growth functions and their vector space analogues. But our analyses of these numbers are quite different. Milne first develops $S_q(n,k)$, offering a combinatorial proof of (2.1), which then plays a central role in his analysis. His approach requires that one exhibit for the vector space case of the map (5.15) a bijection between the sets of preimages associated with any two chains in $\mathscr{B}$.

We first develop $\tilde{S}_q(n,k)$, constructing the explicit formula (5.7) for the number of preimages of any element of $\mathscr{A}$ under the map (5.6). With Theorem 5.2, this entails that every element of $\mathscr{B}$ has the same number of preimages under (5.15). More importantly, (5.7) reveals that $\tilde{S}_q(n,k)$ is a Comtet number, with the immediate consequences (5.9), (5.10), and (5.11). Formula (2.1) is just (5.11) with $x = x_q$.

In the next section we describe q-generalizations of the Lah numbers analogous to $\hat{S}_q(n,k)$, $\tilde{S}_q(n,k)$, and $S_q(n,k)$.

6. The Lah numbers of a q-lattice

The (signless) Lah numbers $L(n,k)$ originated [10] as connection constants:

$$x^{\bar{n}} = \sum_{k=0}^{n} L(n,k)x^{\underline{k}}, \tag{6.1}$$

Clearly, $L(n,0) = \delta_{n,0}$ and $L(0,k) = \delta_{0,k}$ for all $n,k \in \mathbb{N}$. One can also derive from (6.1) in straightforward, if tedious, fashion the recurrence

$$L(n+1,k) = L(n,k-1) + (n+k)L(n,k) \quad \text{for all } n \in \mathbb{N}, k \in \mathbb{P}, \tag{6.2}$$

the formula

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1} \quad \text{for all } n,k \in \mathbb{P}, \tag{6.3}$$

and the generating function

$$\sum_{n \geqslant 0} L(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k \quad \text{for all } k \in \mathbb{N}. \tag{6.4}$$

Since $\binom{n-1}{k-1}$ counts all sequences of k positive integers summing to n, formula (6.3) suggests a simple combinatorial interpretation of the Lah numbers. Just as the Stirling number $S(n,k)$ enumerates the distributions of n labeled balls among k unlabeled urns, with no urn left empty, $L(n,k)$ enumerates such distributions with the added proviso that the balls in each urn are to be linearly ordered [13], [1, pp. 86–87]. Indeed, the best approach to the Lah numbers would be to define them in this way, and derive (6.1) and (6.2) by (easy) combinatorial arguments.

There is an obvious q-lattice generalization of the above. If $[x,y]$ is an n-interval in such a lattice one selects, in all possible ways, from the n_q atoms of $[x,y]$ a subset A of n atoms such that $\vee A = y$, and then distributes these atoms among k unlabeled urns, with no urn left empty, and with the atoms in each urn linearly ordered. By Theorem 3.1 and the combinatorial interpretation of $\binom{n-1}{k-1}$ noted above, there are

$$\hat{L}_q(n,k) := \frac{n_q^{\underline{n}}}{k!}\binom{n-1}{k-1} \tag{6.5}$$

such distributions for $n,k \in \mathbb{P}$.

The numbers $\hat{L}_q(n,k)$ do not generalize $L(n,k)$ in a very profound way, but are included for completeness as obvious analogues of the q-Stirling numbers $\hat{S}_q(n,k)$. A deeper generalization of the Lah numbers requires their reconceptualization along the lines of Milne's restricted growth functions, as described below.

Given n balls, labeled $1,\ldots,n$, and k urns, labelled $1,\ldots,k$, consider the distributions of these balls among these urns, with no urn left empty, and with the balls in each urn linearly ordered. By earlier remarks, there are

$$\lambda(n,k) := k!L(n,k) = n!\binom{n-1}{k-1} \tag{6.6}$$

such distributions.

Associate with each such distribution a sequence

$$((u_1,p_1),\ldots,(u_n,p_n)), \tag{6.7}$$

where u_i denotes the number of the urn in which ball i is placed, and p_i its position in that urn. The sequences arising in this way are characterized by the following two properties:

(i) $\{u_1,\ldots,u_n\} = [k]$, and

(ii) for all $j \in [k]$, with $I_j = \{i \in [n] : u_i = j\}$, $\{p_i : i \in I_j\} = [|I_j|]$.

If S^* denotes the set of sequences satisfying (i) and (ii), then $|S^*| = \lambda(n,k)$.

In each sequence $((u_1,p_1),\ldots,(u_n,p_n))$ the pair $(j,1)$ occurs exactly once for every $j \in [k]$. If $((u_{t_1},1),\ldots,(u_{t_k},1))$ is the subsequence comprised of all such pairs, then, clearly,

$$((u_1,p_1),\ldots,(u_n,p_n)) \mapsto (u_{t_1},\ldots,u_{t_k}) \tag{6.8}$$

is a map from S^* to the set of all permutations of $[k]$.

Moreover, each permutation of $[k]$ has $L(n,k)$ preimages in S^* with respect to the map (6.8). Take, for example, the permutation $(1,2,\dots,k)$. Its preimages (which might be called *Lah restricted growth functions*) are precisely those sequences $((u_i,p_i))_{1\leqslant i\leqslant n}$ in which $(j,1)$ precedes $(j+1,1), j\in[k-1]$, in a left to right scan.

Such sequences correspond to distributions for which the number on the initial ball in urn j is less than the number on the initial ball in urn $j+1$, for all $j\in[k-1]$. But such distributions to *labeled* urns provide obvious canonical representations of the distributions to *unlabeled* urns enumerated by $L(n,k)$. A variant of this argument shows that every permutation of $[k]$ has $L(n,k)$ preimages in S^* with respect to the map (6.8).

The q-Lah numbers $\tilde{L}_q(n,k)$ and $L_q(n,k)$, to be developed in what follows, arise within the same conceptual framework as the q-Stirling numbers $\tilde{S}_q(n,k)$ and $S_q(n,k)$. We have a k-interval $[x,y]$ of a q-lattice, with rank function ρ, and families $\mathscr{S}$, $\mathscr{A}$, and $\mathscr{B}$ defined, as in Section 5, by

$$\mathscr{S}=\{(a_1,\dots,a_n)\text{: each } a_i \text{ is an atom of } [x,y] \text{ and } a_1\vee\cdots\vee a_n=y\}, \tag{6.9}$$

$$\mathscr{A}=\{(\alpha_1,\dots,\alpha_k)\text{: each } \alpha_j \text{ is an atom of } [x,y] \text{ and } \alpha_1\vee\cdots\vee \alpha_k=y\}, \tag{6.10}$$

and

$$\mathscr{B}=\{(\beta_0,\dots,\beta_k)\text{: } x=\beta_0<\beta_1<\cdots<\beta_k=y\}. \tag{6.11}$$

Also as in Section 5 we associate with each $(a_1,\dots,a_n)\in\mathscr{S}$ the sequence $1=i_1<i_2<\cdots<i_k\leqslant n$, where

$$i_j:=\min\{i:\rho(a_1\vee\cdots\vee a_i)=j\}. \tag{6.12}$$

Recall that for all $j\in[k]$, $\rho(\beta_j)=j$, where $\beta_j=a_{i_1}\vee\cdots\vee a_{i_j}$. Thus if $\beta\in[\beta_{j-1},\beta_j]$, then either $\beta=\beta_{j-1}$ or $\beta=\beta_j$. In particular, if α is an atom of $[x,\beta_j]$ then by modularity, $\beta_{j-1}\vee\alpha=\beta_j$ iff $\alpha\nleqslant\beta_{j-1}$.

Now given $(a_1,\dots,a_n)\in\mathscr{S}$ and i_j, as defined by (6.12), we define an ordered partition $(I_1,\dots,I_k)$ of $[n]$, where

$$I_j:=\{i\in[n]\text{: } a_i\leqslant a_{i_1}\vee\cdots\vee a_{i_j}, \text{ but } a_i\nleqslant a_{i_1}\vee\cdots\vee a_{i_{j-1}}\}. \tag{6.13}$$

Note that $i_j\in I_j$ for all $j\in[k]$.

Next, generalizing (6.7), let $\mathscr{S}^*$ be the set of sequences $((a_1,p_1),\dots,(a_n,p_n))$ satisfying (i) $(a_1,\dots,a_n)\in\mathscr{S}$, and (ii) for all $j\in[k]$, $\{p_i:i\in I_j\}=[|I_j|]$.

For every $((a_1,p_1),\dots,(a_n,p_n))\in\mathscr{S}^*$, let $((a_{t_1},1),\dots,(a_{t_k},1))$, with $1\leqslant t_1<t_2<\cdots<t_k\leqslant n$, be the subsequence comprised of the k pairs with second coordinate equal to 1, and consider the map

$$((a_1,p_1),\dots,(a_n,p_n))\mapsto(a_{t_1},\dots,a_{t_k}). \tag{6.14}$$

Theorem 6.1. *The function defined by* (6.14) *maps* $\mathscr{S}^*$ *into* $\mathscr{A}$, *and every element of* $\mathscr{A}$ *has* $\tilde{L}_q(n,k)$ *preimages in* $\mathscr{S}^*$ *with respect to this function, where*

$$\tilde{L}_q(n,k):=\frac{n!}{k!}\binom{n-1}{k-1}_q, \quad \text{for all } n,k\in\mathbb{P}, \tag{6.15}$$

and $\tilde{L}_q(n,0)=\delta_{n,0}$ *and* $\tilde{L}_q(0,k)=\delta_{0,k}$ *for all* $n,k\in\mathbb{N}$.

Proof. While it need not be the case that $t_j \in I_j$, there is some permutation σ of $[k]$ such that $t_{\sigma(j)} \in I_j$. So to show that (6.14) maps $\mathscr{S}^*$ into $\mathscr{A}$, it suffices to show that $a_{t_1} \vee \cdots \vee a_{t_k} = y$ for every sequence $(t_1, \ldots, t_k)$ with $t_j \in I_j$. In fact, it is the case that

$$a_{t_1} \vee \cdots \vee a_{t_j} = a_{i_1} \vee \cdots \vee a_{i_j} \quad \text{for all } j \in [k], \tag{6.16}$$

for all such sequences, by induction on j. The case $j = 1$ of (6.16) holds, since for all $i \in I_1$, $a_i = a_1$. Given (6.16), and $t_{j+1} \in I_{j+1}$, we have

$$a_{i_1} \vee \cdots \vee a_{i_j} \leqslant a_{t_1} \vee \cdots \vee a_{t_{j+1}} \leqslant a_{i_1} \vee \cdots \vee a_{i_{j+1}}, \tag{6.17}$$

and hence $a_{t_1} \vee \cdots \vee a_{t_{j+1}} = a_{i_1} \vee \cdots \vee a_{i_{j+1}}$, since $a_{t_{j+1}} \nleqslant a_{i_1} \vee \cdots \vee a_{i_j}$.

All of the preimages $((a_1, p_1), \ldots, (a_n, p_n))$ of a given $(\alpha_1, \ldots, \alpha_k) \in \mathscr{A}$ under the map (6.14) may be constructed by

(1) choosing a sequence $1 \leqslant t_1 < t_2 < \cdots < t_k \leqslant n$ and setting $(a_{t_j}, p_{t_j}) = (\alpha_j, 1)$ for all $j \in [k]$,

(2) choosing a sequence of nonnegative integers $\delta_1 + \cdots + \delta_k = n - k$,

(3) for each $j \in [k]$, pairing each of the numbers $2, \ldots, \delta_j + 1$ with an arbitrary atom α of $[x, \alpha_1 \vee \cdots \vee \alpha_j]$ for which $\alpha \nleqslant \alpha_1 \vee \cdots \vee \alpha_{j-1}$ and

(4) assigning these $n - k$ pairs as the values of (a_i, p_i) for $i \notin \{t_1, \ldots, t_k\}$.

Hence there are

$$\binom{n}{k} \sum_{\substack{\delta_1 + \cdots + \delta_k = n-k \\ \delta_i \in \mathbb{N}}} (n-k)!(1_q)^{\delta_1}(2_q - 1_q)^{\delta_2} \cdots (k_q - (k-1)_q)^{\delta_k}$$

$$= \frac{n!}{k!} \sum_{\substack{\delta_1 + \cdots + \delta_k = n-k \\ \delta_i \in \mathbb{N}}} q^{0\delta_1 + 1\delta_2 + \cdots + (k-1)\delta_k} = \frac{n!}{k!} \binom{n-1}{k-1}_q \tag{6.18}$$

such preimages, by (1.20) □

If one composes (6.14) with the map $(\alpha_1, \ldots, \alpha_k) \mapsto (x, \alpha_1, \alpha_1 \vee \alpha_2, \ldots, \alpha_1 \vee \cdots \vee \alpha_k)$ from $\mathscr{A}$ to $\mathscr{B}$, then, under the resulting map, each element of $\mathscr{B}$ has

$$L_q(n,k) := q^{\binom{k}{2}} \tilde{L}_q(n,k) \tag{6.19}$$

preimages in $\mathscr{S}^*$, thus completing the analogy with the three types of q-Stirling numbers developed in Sections 4 and 5.

Ironically, in view of the origins of the Lah numbers, none of our q-generalizations of $L(n,k)$ seem to function in any interesting way as connection constants between polynomial sequences. Nor does the recurrence (6.2) generalize in an interesting way.

On the other hand, by (6.18) and (1.17) the generating function (6.4) generalizes nicely for $\tilde{L}_q(n,k)$ to

$$\sum_{n \geqslant 0} \tilde{L}_q(n,k) \frac{x^n}{n!} = \frac{1}{k!} \frac{x^k}{\prod_{0 \leqslant j \leqslant k-1}(1 - q^j x)} \quad \text{for all } k \in \mathbb{N}. \tag{6.20}$$

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References

[1] M. Aigner, Combinatorial Theory (Springer, New York, 1979).
[2] E. Bender and J. Goldman, Enumerative uses of generating functions, Indiana Univ. Math. J. 20 (1971) 753–765.
[3] G. Birkhoff, Lattice Theory, 3rd ed. (American Mathematical Society, Providence, RI, 1967).
[4] L. Carlitz, On Abelian fields, Trans. Amer. Math. Soc. 35 (1933) 122–136.
[5] L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948) 987–1000.
[6] L. Comtet, Nombres de Stirling généraux et fonctions symmetriques, C.R. Acad. Sci. Paris Série A, 275 (1972) 747–750.
[7] R. Davis, Covers, q-binomial series, and q-lattices, dissertation, Univ. Tennessee, 1991.
[8] R. Davis and C. Wagner, Covering algebras and q-binomial generating functions, Discrete Math. 128 (1994) 95–111.
[9] P. Doubilet, G.-C. Rota and R. Stanley, On the foundations of combinatorial theory (VI): the idea of generating function, Sixth Berkeley Symp. on Math. Statist. and Probab., Vol. 2: Probability Theory, Univ. of California (1972) 267–318.
[10] I. Lah, Eine neue art von zahlen, ihre eigenschaften und anwendung in der mathematischen statistik, Mitteilungsbl, Math. Statist. 7 (1955) 203–212.
[11] S. Milne, A q-analogue of restricted growth functions, Dobinski's inequality, and Charlier polynomials, Trans. Amer. Math. Soc. 245 (1978) 89–118.
[12] S. Milne, Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers, Adv. Math. 43 (1982) 173–196.
[13] T.S. Motzkin, Sorting numbers for cylinders and other classification numbers, Proc. Symp. Pure Math., Vol. 19 (American Mathematical Society, Providence, RI, 1971) 167–176.
[14] R. Stanley, Enumerative Combinatorics, Vol. I (Wadsworth and Brooks/Cole, Monterey, 1986).
[15] M. Wachs and D. White, p, q Stirling numbers and partition statistics, J. Combin. Theory A 56 (1991) 27–46.
[16] C. Wagner, Surjections, differences, and binomial lattices, Stud. Appl. Math. 93 (1994) 15–27.
[17] M. Ward, A calculus of sequences, Amer. J. Math. 58 (1936) 255–266.