One-dimensional group construction

Dustin Cartwright

University of Tennessee, Knoxville

July 10, 2025

Algebraic matroids

- K: a field (of any characteristic)
- C: a one-dimensional irreducible variety over K
- $X \subset C^n$: irreducible subvariety
- $\pi_I : C^n \to C^I$ coordinate projection onto the coordinates $I \subset [n]$
- Matroid of X:
    - Independent sets are I such that $\pi_I(X) \subset C^I$ is dense
    - Bases are sets I such that $\pi_I|_X$ is dominant and generically finite.
    - Rank of I is the dimension of $\overline{\pi_I(X)}$

Matrix of endomorphisms

- $C = G$: is a 1-dimensional connected algebraic group (i.e. $(K, +)$, $(K^\times, \times)$, elliptic curve)
- M: $n \times r$ matrix with $M_{ij}\colon G \to G$: group endomorphisms of G

$$\Phi\colon G^r \to G^n$$
$$\Phi_i(g_1, \ldots, g_r) = M_{i1}(g_1) \cdots M_{ir}(g_r)$$

Take $X = \phi(G^r) \subset G^n$.

Special case: linear matroids

Take $G = (K, +)$. For $a \in K$, can consider a as an endomorphism defined by $a(g) = ag$.
Then,

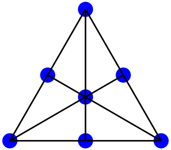
$$\Phi_i(g_1, \ldots, g_r) = M_{i1}g_1 + \cdots + M_{ir}g_r$$

$X = \Phi(\mathbb{A}^r)$ is the column space of M, with the independent sets the set of linearly independent rows

Special case: toric varieties

Take $G = (K^\times, \times)$. Write $n \in \mathbb{Z}$ for the endomorphism $n(g) = g^n$.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \Phi(x, y, z) = \begin{pmatrix} x \\ y \\ z \\ xy \\ xz \\ yz \\ xyz \end{pmatrix}$$



$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$ is in the left null space, corresponding to $(x)(y)(xy)^{-1} - 1$ gives a dependence

One-dimensional group construction

$\mathbb{E}$ is the endomorphism ring of a 1-dimensional connected algebraic group G. $\mathbb{E}$ is contained in a division ring Q.

- M is an $n \times r$ matrix with entries in $\mathbb{E}$
- $\Phi \colon G^r \to G^n$ is the group homomorphism defined by M
- The algebraic matroid of $X = \Phi(G^r)$ is the same as the row span of M

(Lindström 1988, Evans-Hrushovski 1991)

One-dimensional algebraic groups

- $G = (K, +)$, $\mathbb{E} = Q = K$ in characteristic 0, $\mathbb{E} = K[F]$ in characteristic p, skew polynomial ring with $F\alpha = \alpha^p F$ for $\alpha \in K$
- $G = (K^\times, \times)$: $\mathbb{E} = \mathbb{Z}$, $Q = \mathbb{Q}$
- G an elliptic curve: Q is $\mathbb{Q}$, imaginary quadratic extension, or quaternion algebra over $\mathbb{Q}$ (if $p > 0$), $\mathbb{E}$ is finitely generated subring of Q.

In all cases, there is a $\mathbb{Z}$-valuation on Q with commutative residue field.

Non-commutative example

K has characteristic p, $G = (K, +)$, $\mathbb{E} = K[F]$
Pick $\alpha \in K$

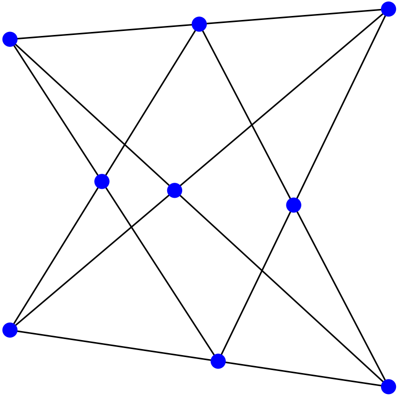
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \alpha \\ 1 & F \end{pmatrix}$$

Then X is the image of:

$$\Phi(x, y) = (x, y, x + \alpha y, x + y^p)$$

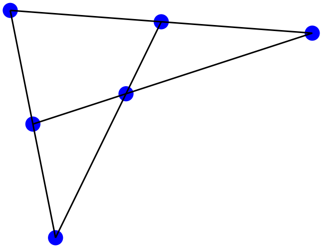
$\begin{pmatrix} F - \alpha^p & 0 & -F & \alpha^p \end{pmatrix}$ is in the left null space, gives
$(x^p - \alpha^p x) - (x + \alpha y)^p + \alpha^p (x + y^p) = 0$ as a dependency

Non-Pappus matroid



The non-Pappus matroid is linear over any non-commutative division ring and therefore algebraic over any field of positive characterisitic.

Evans-Hrushovski theorem



Theorem (Evans-Hrushovski 1991)

Any $X \subset C^6$ which whose matroid is the graphic matroid of K_4 is equivalent to a one-dimensional group construction $G^3 \to G^6$.

As a corollary, they establish a Mnëv-type theorem.

Characteristic sets

The algebraic characteristic set $\chi(M)$ of a matroid M is the subset of $\{0\} \cup \{\text{primes}\}$ such that there exists a subvariety whose algebraic matroid is M.

Theorem (C.-Varghese 2024)

Let C be either a finite or cofinite subset of $\{0\} \cup \{\text{primes}\}$.
Assume that if $0 \in C$, then C is cofinite.
Then there exists a matroid M such that $\chi(M) = C$.

Density

Theorem (C.-Varghese 2024)

Let $0 \leq \alpha < \beta \leq 1$. Then there exists a matroid M such that:

$$\alpha < \lim_{N->\infty} \frac{|\{p \in \chi_A(M), p < N\}|}{|\{p \text{ prime}, p < N\}|} < \beta$$

The densities of algebraic characteristic sets are dense.

Question

Can $\chi(M)$ be infinite and have density 0? Non-cofinite and density 1?