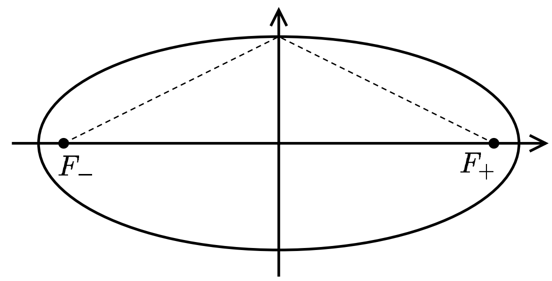
Here is a project problem for extra credit that blends sophomore ODEs with some good, classical geometry. As a problem, it caters primarily to mathematics majors, but in its contents it will impart some useful knowledge for physics and electrical engineering majors, too. And it will help deepen mastery of calculus for any major.

All the remarks in oblique font give background information that you do not need for doing the problem, but which help appreciate the problem in its general context.



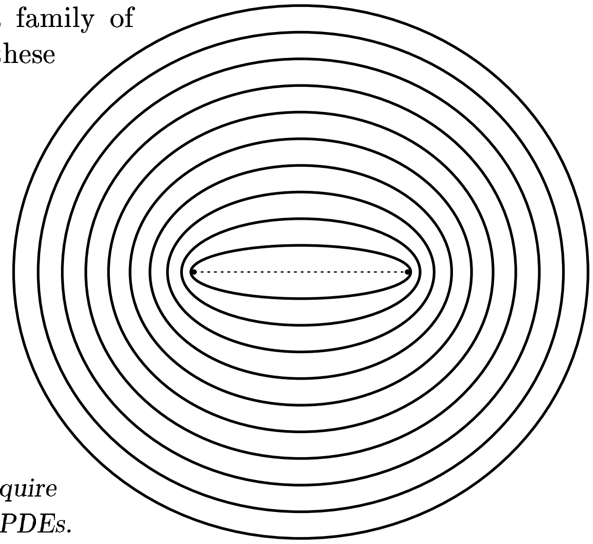
An ellipse is a curve that can be described in cartesian coordinates as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (The special case $a = b = r$ produces a circle of radius r.) We will deal here with a family of ellipses where b is arbitrary, but $a = \sqrt{b^2 + 1}$.

In the picture we assume that $a > b$. So horizontally the ellipse extends from $x = -a$ to $x = a$, which is why we call a the (major) semiaxis of the ellipse. Vertically it extends from $y = -b$ to $y = b$, which is why we call b the (minor) semiaxis of the ellipse. The two points $(\pm\sqrt{a^2 - b^2}, 0) =: (\pm e, 0)$ play a special role, they are called the foci F_- and F_+ of the ellipse. (Foci is the plural of 'focus'). You can easily see that each of the dashed segments in the figure has length a.

The name focus stems from the fact that if you think of the ellipse as a perfect mirror with a light source in one focus, then the reflected rays will meet in the other focus. If you make one focus into the position of a TV sattellite, and the other focus into the point where the receiver that comes with your satellite dish is sitting, then your satellite dish should have the curved shape of a tiny piece of an ellipse (rotated around the axis satellite–receiver). With such large distances between the foci however, and such a tiny satellite dish (tiny compared to the huge distance between the satellite and your dish), the cross section of your satellite dish will be indistinguishable from a parabola, so the manufacturer is perfectly justified in calling it a paraboloid rather than an ellipsoid.

There are many different ways of describing an ellipse. Here is how the gardener designs an elliptic flower bed: Put poles in the ground at the position of the foci. Tie a rope around the poles and pull it tight. Like the dotted line in the figure. Move a marker along the ground, always keeping the rope pulled tight with this marker. It will describe an ellipse. In other words, each point on the ellipse is such that the sum of its distances to F_- and to F_+ is $2a$. If you write this property in formulas, namely $\sqrt{(x+e)^2 + y^2} + \sqrt{(x-e)^2 + y^2} = 2a$ and simplify diligently and patiently, you should indeed get the equation for an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{a^2 - e^2} = 1$.

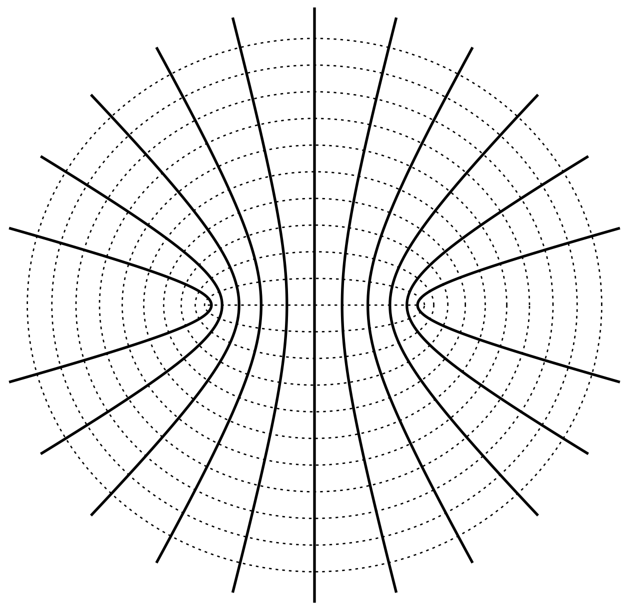
As promised before, we are now considering a family of ellipses where $a^2 = b^2 + 1$, i.e., $e = 1$. So all of these ellipses will have the same foci in common, and we call it a confocal family of ellipses. Here is a picture:



The picture has an electrotechnical interpretation: If you think of the dotted line as the cross section of a thin long metal strip extending vertically out of the paper and behind the paper and put a certain voltage on this strip, then the ellipses will be lines of constant potential (voltage difference with respect to the metal strip). The reason why this electrostatic interpretation leads to the drawn ellipses is withheld here, since it would require theoretical physics beyond the sophomore level, and PDEs.

Problem: Find those curves that will always intersect the family of confocal ellipses in a right angle.

Answer: These curves will be hyperbolas, and they, too will share the same foci F_- and F_+. (Now if you don't know much about hyperbolas yet, this will not tell you much, but your job is anyways to calculate that whole thing, not to take the answer by faith, so you will learn to understand the answer as you go.) Anyways, here is a picture to illustrate the situation: The ellipses are now dotted, and the curves you are looking for are solid.



Solution Guide: Fix e, the parameter determining the foci. Pretend that the family of ellipses makes up the general solution of some 1st order ODE, where b is the free parameter (or constant of integration). You have to find the ODE from the family of solutions (just the opposite of what you are usually doing!) This is done by determining the direction field:

(a) For each curve $\frac{x^2}{e^2+b^2} + \frac{y^2}{b^2} = 1$, you determine its slope at any point by implicit differentiation.
(b) to find the direction (slope) at any point (x, y), as a function of x, y only, you first need to know on which curve it lies, i.e., you need to find b from $\frac{x^2}{e^2+b^2} + \frac{y^2}{b^2} = 1$, with (x, y) given. That b gets plugged into the (implicit) formula for the derivative y'. — Practically, you have to eliminate b from the equation describing the family of ellipses, and its derivative, and you start eliminating b from where it is conveninent.
(c) Now you have a relation involving only x, y, y' (and e), but not b. It is the ODE whose solutions are graphed as ellipses from the original family. From there you can find a **new** ODE, with a **new** direction field, namely all new directions are orthogonal to the corresponding old directions. You must know or look up the answer to the following question: If a line has slope m, then lines orthogonal to it have which slope? — This said, in the old ODE, you replace y' with what expression to get the new ODE?
(d) Simplify the new ODE and compare it with the old ODE. If your first impression is 'This cannot possibly be true, something must be badly wrong', then you are probably perfectly right. You will be able to solve the new ODE, but nevertheless you must most likely come in for an individual discussion with me to resolve the apparent paradox. And you will have found the family of orthogonal curves to the confocal ellipse family.