Homework Chapter 5
UTK – M251 – Matrix Algebra
Fall 2003, Jochen Denzler, MWF 1:25–2:15, Ayres 318

1. Check if the following set of vectors in $\mathbb{R}^4$ is linearly independent. Show your work. $S = \{[0,\ 1,\ 2,\ 3]^T,\ [1,\ 2,\ 3,\ 4]^T,\ [0,\ 2,\ 0,\ 2]^T\}$

2. Explain in one sentence why it can be seen at a glance that the set in the previous example is not a basis of $\mathbb{R}^4$.

3. Is $S = \{[1,\ 1,\ 1]^T,\ [2,\ -2,\ 1]^T,\ [4,\ 0,\ 3]^T\}$
(a) linearly independent?
(b) a spanning set for $\mathbb{R}^3$?
(c) a basis for $\mathbb{R}^3$?

Each answer requires an explanation.

4. Show that $S = \{[1,\ 1,\ 1]^T,\ [2,\ -2,\ 1]^T,\ [1,\ 0,\ 1]^T\}$ is a basis of $\mathbb{R}^3$.

5. Show that the set $S = \{x^2+1, 2x, 2x^2-3\}$ is a basis of the vector space P_2 of polynomials of degree at most 2.

6. Is the set $S = \{1, \sin x, \cos x\}$ linearly independent in the vector space $C^0(\mathbb{R})$ of continuous real functions? – Hint: You have to address the question: If $k_1 1 + k_2 \sin x + k_3 \cos x = \mathbf{0}$ (this $\mathbf{0}$ means the zero *function*), in other words, if $k_1 1 + k_2 \sin x + k_3 \cos x = 0$ for *every* x, can we conclude that $k_1 = k_2 = k_3 = 0$?

7. Is the set $S = \{1, \sin^2 x, \cos^2 x\}$ linearly independent in the vector space $C^0(\mathbb{R})$ of continuous real functions?

8. Find the coordinate vector of the vector $[1,\ 2,\ 3]^T$ with respect to the basis $S = \{[1,\ 1,\ 1]^T,\ [2,\ -2,\ 1]^T,\ [1,\ 0,\ 1]^T\}$ of $\mathbb{R}^3$ from pblm. 4.

9. Find the coordinate vector of the polynomial $x^2 + x + 1$ with respect to the basis $S = \{x^2 + 1, 2x, 2x^2 - 3\}$ from pblm. 5.

10. Show that if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are non-zero vectors in $\mathbb{R}^3$ that are mutually orthogonal, then $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is linearly independent.

11. According to Thm. 5.2.2 in the book, the set of solutions to a *homogeneous* linear system in n unknowns is a linear subspace of $\mathbb{R}^n$, called the solution space of the linear system.

Find a basis for the solution space of the homogeneous linear system, and determine the dimension of the solution space.

$$\begin{aligned} 3x_1 - x_2 - x_3 + 3x_4 &= 0 \\ 4x_1 + x_2 - x_3 + 2x_4 &= 0 \end{aligned}$$

12. Same question for the system

$$\begin{aligned} 3x_1 - x_2 - x_3 + 3x_4 &= 0 \\ 4x_1 + x_2 - x_3 + 2x_4 &= 0 \\ x_1 + x_2 + x_3 - 4x_4 &= 0 \end{aligned}$$

13. The real vector space $C^2(\mathbb{R})$ contains all twice differentiable real functions whose second derviative is still continuous. In M231 you have learned (or will learn) that a function $y = f(x)$ satisfies the condition $f''(x) - 3f'(x) + 2f(x) = 0$ for all x, if and only if f is of the form $f(x) = c_1 e^x + c_2 e^{2x}$. You may just take this fact for granted in this class, and now this should be an *easy* problem:

Give a basis for the vector space W consisting of those functions that satisfy $f''(x) - 3f'(x) + 2f(x) = 0$ for all x; and determine the dimension of W.

14. What is the dimension of $\mathbb{R}^{3\times 3}$ (also known under the name M_{33}, the vector space of all 3×3 matrices? Give two different bases for this vector space.

15. $S_0^{3\times 3}$ is the set of all *symmetric* 3×3 matrices whose trace is 0. Make sure that you understand that $S_0^{3\times 3}$ is a subspace of $\mathbb{R}^{3\times 3}$. What is the dimension of $S_0^{3\times 3}$? Give a basis. (If you have forgotten the definition of the trace, it's early in the glossary, or use the index in the book.)

16. Draw into one single picture of the plane $\mathbb{R}^2$ the following objects:

(a) The solution set of the *homogeneous* linear system $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (with each solution being represented as a *point* in $\mathbb{R}^2$).

(b) The solution set of the *inhomogeneous* linear system $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$ (with each solution being represented as a *point* in $\mathbb{R}^2$).

(c) For two particular solutions of the inhomogenous system from part (b), according to your free choice, highlight the point representing them in color, and also draw the respective *vectors* from the origin to these points in the same color.

I want a decent picture that you would not be ashamed to xerox for your class if you were the teacher.

17. Study sections 5.5 and 5.6 as needed; let

$$A = \begin{bmatrix} 3 & 2 & 1 & -1 & 4 \\ 5 & 8 & 7 & 3 & 6 \\ -2 & 1 & 2 & 3 & -3 \end{bmatrix}$$

Invent one non-zero vector that is in the row space of A, but is not among the rows of A. Also determine the dimension of the row space of A by giving an explicit basis for it.

18. Same matrix A as before. Determine a basis for the column space of A. (There are many correct solutions of course.) Give the dimension of this column space.

19. Same matrix A as before. Is the vector $[1, 1, 1]^T$ in the column space of the matrix A? Show your work. (No credit for merely saying yes/no.)

20. Same matrix A as before. Determine the null space of A.

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21. Draw the vectors $\boldsymbol{u} = [2, 1]^T$ and $\boldsymbol{v} = [-1, 1]^T$ in the plane. (Again we need a decent figure that you would not be ashamed to xerox for your class if you were the teacher. A unit length of 1 or 1.5 inches is perfectly appropriate.) First rotate each of them by 60 degrees counterclockwise (geometrically, using ruler and protractor), then reflect the resulting vectors in the line $x_2 = -2x_1$. You should have one line and six vectors in your figure now. In a different color, I want you to start over with $\boldsymbol{u}$ and $\boldsymbol{v}$, but this time first reflect them in the line $x_2 = -2x_1$, and then rotate the result counterclockwise by 60 degrees. That will add 4 more vectors in a different color to the figure.

22. Same $\boldsymbol{u}$ and $\boldsymbol{v}$ as before. Refer back to the figure from the previous problem.

Find the matrix Q that represents the rotation by 60 degrees counterclockwise, and the matrix S that represents the reflection in the line $x_2 = -2x_1$. (You had one problem of this type in Chapter 4 already and may refer back for help.) Calculate both products QS and SQ. Label all vectors in the previous problem with $\boldsymbol{u}$, $\boldsymbol{v}$, $Q\boldsymbol{u}$, $S\boldsymbol{v}$, $QS\boldsymbol{u}$, etc., as appropriate. Calculate the vectors $QS\boldsymbol{u}$ and $SQ\boldsymbol{u}$ algebraically and compare with the figure.

23. Calculate the determinant

$$\begin{vmatrix} a & 2 & 3 & b \\ 1 & -1 & 2 & 5 \\ 2 & 2 & -1 & c \\ d & 2 & 3 & -2 \end{vmatrix}$$

in a reasonably efficient way using at least one row or column operation and at least one cofactor expansion of some row or column.