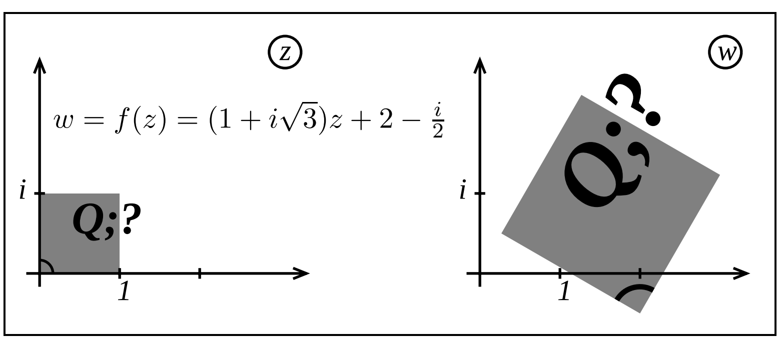
Examples of mappings defined by certain complex functions

Before we continue with more theory, I want you to experience a few examples of simple complex functions. Just as you would have graphed linear functions, quadratics, and similar simple functions in precalculus before venturing into single variable calculus, we will graph complex functions here. But for a complex function $w = f(z)$, you cannot plot a diagram with a z and a w axis, because actually you would need a z- and a w-*plane*! So the whole picture would need not 2, but 4 dimensions! So we invent a different method for graphing functions. Namely, we draw a z-plane and a w-plane separately. Then we draw some nice geometric pattern in the z-plane; and to each point z in this pattern we plot the corresponding point $w = f(z)$ in the w-plane. This way, a new pattern arises in the w-plane. Just like in optical imaging, f describes some kind of imaging process, and understanding this imaging process geometrically is our way of 'graphing' f.

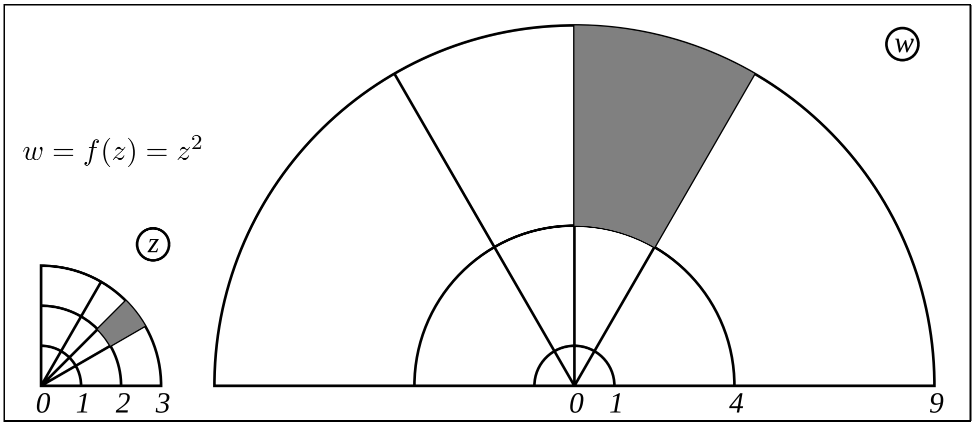
This process is transparent only for rather simple functions f, and a judiciously selected sampling of patterns for each function f. For a complicated function and some random pattern, the pictures you'd get may well be a less-than-enlightening mess. But the good news is, that these few simple examples are actually quite useful in applications.

All of the functions we are studying here will later turn out to be *differentiable*. All the mappings you will encounter will turn out to be *conformal*, i.e., angles in the image equal corresponding angles in the original. (There are a few exceptional points, can you spot them?) This is no coincidence. Differentiable functions give rise to conformal mappings (subject to some details I am skipping here; details to follow later). In 2-dimensional electrostatics problems, this conformal mapping property is desired, hence the applicability of complex variables to the subject.

We start our discussion with a simple example, a *linear* function. Say $f(z) = (1+i\sqrt{3})z + 2 - \frac{i}{2}$. Note that in polar coordinates, $1 + i\sqrt{3} = 2\operatorname{cis}(\frac{\pi}{3})$. So what is f doing to z? It doubles the absolute value of z and adds $\frac{\pi}{3}$ to its argument. Then it translates the result by $2 - \frac{i}{2}$, which is the real vector $[2, -\frac{1}{2}]^T$. The pattern I have selected here, the gray square with random text in it, gets enlarged by a factor $2 = |1 + i\sqrt{3}|$, rotated by an angle $\frac{\pi}{3} = \arg 1 + i\sqrt{3}$, and translated by $2 - \frac{i}{2}$. Any other pattern would undergo the same changes.

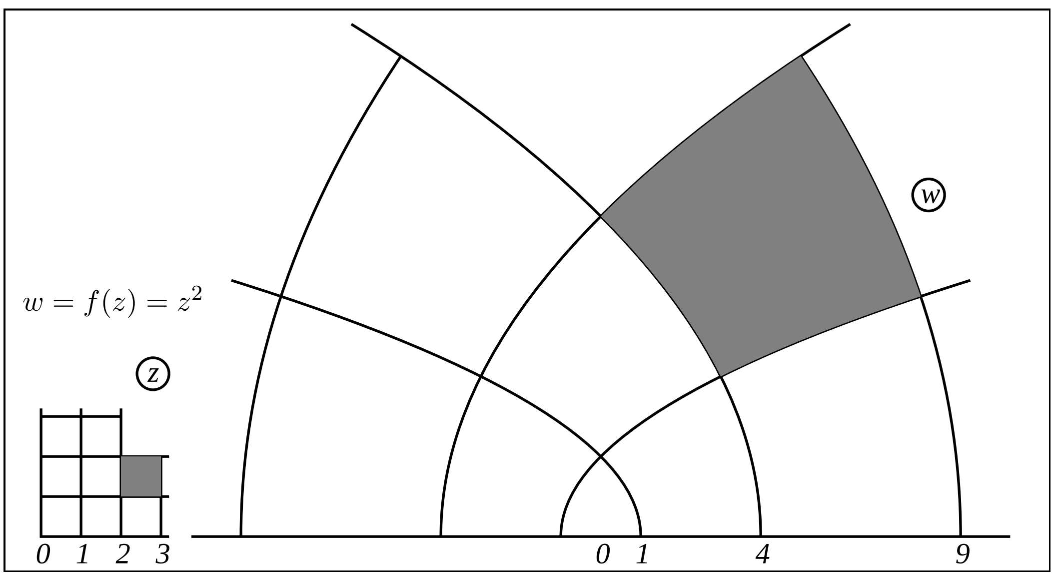


Now let's have a look at the mapping $w = f(z) = z^2$. Here it's nice to use certain cartesian or curvilinear coordinate grids as patterns. Remember: squaring a complex number squares the absolute value and doubles the argument. So it is easy to understand the mapping for a piece of a polar coordinate grid.



Another example: Still the same function $f(z) = z^2$, but we map a cartesian coordinate grid: We study what happens to the straight vertical lines $z = 1 + it$, $z = 2 + it$, $z = 3 + it$ under the mapping $w = z^2$. Then we study the same for the horizontal lines $z = i + t$, $z = 2i + t$, $z = 3i + t$.

For instance if $z = 1 + it$, we get $w = z^2 = 1 - t^2 + 2it$. So if we write $w = u + iv$, we observe that $u = 1 - t^2 = 1 - v^2/4$. This is a parabola that opens to the left. Similar parabolas are obtained for the other vertical lines. For the horizontal lines like eg., $z = i + t$, we get $w = z^2 = (i + t)^2 = t^2 - 1 + 2it = u + iv$, so $u = v^2/4 - 1$, which is a parabola that opens to the right. I am drawing all parabolas in a to-scale figure:



Did you notice how angles are preserved? Right angles in the original will still be right angles in the image (and this would apply to other angles as well). There is however one exceptional point z_0. Angles of lines meeting at z_0 are NOT the same as angles between the image lines meeting at $f(z_0) = z_0^2$. Can you spot this point?

Hwk 7: *Consider the curves* $xy = const$ *and* $x^2 - y^2 = const$*, say in the first quadrant of* $\mathbb{C}$*. These curve are hyperbolas. If these hyperbolas are mapped under the function* $f(z) = z^2$*, what curves arise in the complex* w*-plane?*

Hwk 8: *Now consider the map* $f(z) = 1/z$*. Using the problem pg 12,#25, show that*
• *every circle in the* z *plane that does not pass through 0 is mapped into another circle in the* w *plane (that also doesn't pass through 0);*
• *every straight line that does not pass through 0 is mapped into a circle in the* w *plane that does pass through 0;*
• *every circle in the* z *plane that does pass through 0 is mapped into a straight line in the* w *plane that doesn't pass through 0;*
• *every straight line in the* z *plane that passes through 0 is mapped into an other straight line in the* w *plane that again passes through 0.*

The circle centered at 2 *with radius 1 is mapped into which circle? Notice that this image circle is* NOT *centered at* $f(2) = \frac{1}{2}$*.*

Draw a cartesian coordiante grid involving the vertical lines $\operatorname{Re} z \in \{-3, -2, -1, 0, 1, 2, 3\}$ *and also the horizontal lines* $\operatorname{Im} z \in \{-3, -2, -1, 0, 1, 2, 3\}$*. What are the images of this grid under the mapping* $w = f(z)$ *in the* w *plane? Draw a picture.*

Hwk 9: *The function* $f(z) = z + \frac{1}{z}$ *is an interesting example. Show that all circles* $|z| = r$ *for* $r > 1$ *map into ellipses* $w = u + iv$ *where* $\frac{u^2}{?^2} + \frac{v^2}{?^2} = 1$*. (Fill in the question marks with appropriate expressions in* r*). In particular conclude that the exterior of the unit circle, namely the set* $|z| > 1$ *maps into the set* $\mathbb{C} \setminus [-2, 2]$*, and that this map is a 1-1 correspondence (in other words bijective).*

Hwk 10: *Consider the function* $f(x + iy) = e^x \operatorname{cis} y$*. [We will later identify this function as* $f(z) = e^z$*, as you may be aware already from Math 231]. What does the rectangle* $a < x < b$*,* $c < y < d$ *map into? What kind of curve does a line* $z = (1 + 3i)t$ *map into? Sketch.*

This probably comes close to exhausting the 'good' examples that are accessible at this stage.